Writing Formal Language Theory Proofs: Finite-State Automata Example

11-711: Algorithms for NLP

August 29, 2014
Writing Formal Proofs
Proof Writing

Formal proof:

Starting from basic axioms (definitions), demonstrate step by step that a statement is necessarily true. Using proper theorems may make your life much easier.

Some hints:

- Not just trying to convince the reader that you know what you’re talking about

- The step-by-step logic behind the proof is the most important.

A great reference: Alon’s proof on the equivalence of DFSA and NDFSA.
A possible template for FSA(T)-related proof

1. Check if you need to prove in two directions. Don’t miss the easy one.
2. A precise description of your construction. A figure is good, but not enough.
3. Prove your construction is correct. Prove by induction may be helpful in this step. (by the length of sequence, by the number of states, etc.)
4. Explicitly write out what you just proved and why.
FSA Proof Examples
Dual Control Finite State Automaton

FSA

\[ q_0 \xrightarrow{a} q_1 \]

One input
Two machines
Dual Control Finite State Automaton

DCFSA

q0 -> a -> q1

One input
Two machines
Dual Control Finite State Automaton

DCFSA

One input

Two machines
Dual Control Finite State Automaton

FSA $A = (Q, \Sigma, \delta, q_0, F)$
DCFSA $A = (Q_1, Q_2, \Sigma, \delta_1, \delta_2, q^1_0, q^2_0, F_1, F_2)$

- One alphabet
- Two independent sets of states (including initial and final)
- Two independent transition functions
- For each input symbol, both controls change state
- **Only** accepts if both controls are in final state at end of input
Dual Control Finite State Automaton

A_1

\[ q_0 \xrightarrow{a} q_1 \]

Regular?
Dual Control Finite State Automaton

A_1

\[ q_0 \xrightarrow{a} q_1 \]

\[ a a^* \]
Dual Control Finite State Automaton

\[ A_1 \]
\[ A_2 \]

\[ q_0 \quad a \quad q_1 \]
\[ q_0 \quad a \quad q_1 \quad a \quad q_2 \]

\[ aa^* \]
Dual Control Finite State Automaton

\[ A_1 \]

\[ A_2 \]

- For \( A_1 \): \( q_0 \) to \( q_1 \) on input \( a \).
- For \( A_2 \): \( q_0 \) to \( q_1 \) to \( q_2 \) on input \( a \).

Strings accepted: \( aa^* \) for \( A_1 \) and \( aaa^* \) for \( A_2 \).
Dual Control Finite State Automaton

\[ A_1 \]

\[ q_0 \xrightarrow{a} q_1 \]

\[ aa^* \cap aaa^* \]

\[ A_2 \]

\[ q_0 \xrightarrow{a} q_1 \xrightarrow{a} q_2 \]
Dual Control Finite State Automaton

\[ A_1 \]

\[ q_0 \xrightarrow{a} q_1 \]

\[ \text{aa}^* \cap \text{aaa}^* \]

Regular?

\[ A_2 \]

\[ q_0 \xrightarrow{a} q_1 \xrightarrow{a} q_2 \]
FSA-DCFSA Equivalence

Prove that the set of languages accepted by a Dual Control Finite State Automaton is regular.
FSA-DCFSA Equivalence

Prove that the set of languages accepted by a Dual Control Finite State Automaton is regular.
FSA has the same representation ability of regular language.
For DCFSA A, construct FSA A' that accepts the same language
Part 1: FST Construction

Define your construction.

DCFSA $A = (Q_1, Q_2, \Sigma, \delta_1, \delta_2, q_0^1, q_0^2, F_1, F_2)$
Part 1: FST Construction

Define your construction.

DCFSA $A = (Q_1, Q_2, \Sigma, \delta_1, \delta_2, q_0^1, q_0^2, F_1, F_2)$

FSA $A' = (Q', \Sigma', \delta', q'_0, F')$

$Q' = ?$
$\Sigma' = ?$
$\delta' = ?$
$q'_0 = ?$
$F' = ?$
Part 1: FST Construction

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DCFSA $A = (Q_1, Q_2, \Sigma, \delta_1, \delta_2, q_0^1, q_0^2, F_1, F_2)$
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$Q'$ =
Part 1: FST Construction

Define your construction.

DCFSA $A = (Q_1, Q_2, \Sigma, \delta_1, \delta_2, q_0^1, q_0^2, F_1, F_2)$

FSA $A' = (Q', \Sigma', \delta', q_0', F')$

$Q' = \{[q_1, q_2] \mid q_1 \in Q_1, q_2 \in Q_2\}$

$\Sigma' =$
Define your construction.

DCFSA $A = (Q_1, Q_2, \Sigma, \delta_1, \delta_2, q_0^1, q_0^2, F_1, F_2)$

FSA $A' = (Q', \Sigma', \delta', q_0', F')$

\[
Q' = \{ [q_1, q_2] \mid q_1 \in Q_1, q_2 \in Q_2 \}
\]
\[
\Sigma' = \Sigma
\]
\[
\delta' :=
\]
Part 1: FST Construction

Define your construction.
DCFSA $A = (Q_1, Q_2, \Sigma, \delta_1, \delta_2, q_0^1, q_0^2, F_1, F_2)$

FSA $A' = (Q', \Sigma', \delta', q_0', F')$

$Q' = \{[q_1, q_2] \mid q_1 \in Q_1, q_2 \in Q_2\}$

$\Sigma' = \Sigma$

$\delta' := \delta'([q_1, q_2], a) = [\delta_1(q_1, a), \delta_2(q_2, a)]$

$q_0' =$
Part 1: FST Construction

Define your construction.

DCFSA $A = (Q_1, Q_2, \Sigma, \delta_1, \delta_2, q^1_0, q^2_0, F_1, F_2)$

FSA $A' = (Q', \Sigma', \delta', q'_0, F')$

\[
\begin{align*}
Q' &= \{[q_1, q_2] \mid q_1 \in Q_1, q_2 \in Q_2\} \\
\Sigma' &= \Sigma \\
\delta' &= \delta'([q_1, q_2], a) = [\delta_1(q_1, a), \delta_2(q_2, a)] \\
q'_0 &= [q^1_0, q^2_0] \\
F' &=
\end{align*}
\]
Part 1: FST Construction

Define your construction.

DCFSA $A = (Q_1, Q_2, \Sigma, \delta_1, \delta_2, q_0^1, q_0^2, F_1, F_2)$

FSA $A' = (Q', \Sigma', \delta', q_0', F')$

\[
Q' = \{[q_1, q_2] \mid q_1 \in Q_1, q_2 \in Q_2\}
\]

\[
\Sigma' = \Sigma
\]

\[
\delta' := \delta'([q_1, q_2], a) = [\delta_1(q_1, a), \delta_2(q_2, a)]
\]

\[
q_0' = [q_0^1, q_0^2]
\]

\[
F' = \{[f_1, f_2] \mid f_1 \in F_1, f_2 \in F_2\}
\]
Part 2: Correctness of Construction

In order to prove $A$ and $A'$ accepts the same language, we want to show that $w \in L(A')$ if and only if $w \in L(A)$:

$$\hat{\delta}'(q_0', w) = [p_1, p_2] \iff \left( \hat{\delta}_1(q_0^1, w) = p_1 \right) \land \left( \hat{\delta}_2(q_0^2, w) = p_2 \right)$$

Induction on length of $w$
**Part 2: Proof by Induction**

Defined: $w$, $\hat{\delta}$, $(Q', \Sigma', \delta', q'_0, F')$, $(Q_1, Q_2, \Sigma, \delta_1, \delta_2, q^1_0, q^2_0, F_1, F_2)$

**Base:**

$|w| = 0 \iff w = \epsilon$

\[
\hat{\delta}'(q'_0, \epsilon) = [q^1_0, q^2_0] \iff \left( \hat{\delta}_1(q^1_0, \epsilon) = q^1_0 \right) \land \left( \hat{\delta}_2(q^2_0, \epsilon) = q^2_0 \right) \tag{1}
\]

by sub. $\epsilon$ and def. of $\delta'$, $\delta_1$, $\delta_2$

Proved for $|w| = 0$
Part 2: Proof by Induction

Defined: \( w, \hat{\delta}, (Q', \Sigma', \delta', q'_0, F') \), \((Q_1, Q_2, \Sigma, \delta_1, \delta_2, q_1^0, q_2^0, F_1, F_2)\)

Induction: \(|w| = n + 1\)

Split \( w = xa \), where \(|x| = n\)
Part 2: Proof by Induction

Defined: \( w, \hat{\delta}, (Q', \Sigma', \delta', q'_0, F') \), \((Q_1, Q_2, \Sigma, \delta_1, \delta_2, q^1_0, q^2_0, F_1, F_2)\)

**Induction:** \(|w| = n + 1\)

Split \( w = xa \), where \(|x| = n\)

\[ \hat{\delta}'(q'_0, w) = \hat{\delta}'(q'_0, xa) \] \hspace{1cm} (2) by definition of \( w \)
Part 2: Proof by Induction

Defined: \( w, \hat{\delta}, (Q', \Sigma', \delta', q'_0, F'), (Q_1, Q_2, \Sigma, \delta_1, \delta_2, q^1_0, q^2_0, F_1, F_2) \)

Induction: \( |w| = n + 1 \)

Split \( w = xa \), where \( |x| = n \)

\( \hat{\delta}'(q'_0, w) = \hat{\delta}'(q'_0, xa) \)  \( \text{For } n = 0, \ |w| = 1 \text{ so } x = \epsilon \)  \( \text{(2)} \)

by definition of \( w \)
Part 2: Proof by Induction

Defined: $w$, $\hat{\delta}$, $(Q', \Sigma', \delta', q'_0, F')$, $(Q_1, Q_2, \Sigma, \delta_1, \delta_2, q_0^1, q_0^2, F_1, F_2)$

**Induction:** $|w| = n + 1$

Split $w = xa$, where $|x| = n$

$\hat{\delta}'(q'_0, w) = \hat{\delta}'(q'_0, xa)$

For $n = 0$, $|w| = 1$ so $x = \epsilon$

(2) by definition of $w$

$$\hat{\delta}'(q'_0, x) = [p_1, p_2] \iff (\hat{\delta}_1(q_0^1, x) = p_1) \land (\hat{\delta}_2(q_0^2, x) = p_2)$$

(3) by inductive hyp.
Part 2: Proof by Induction

Defined: $w, \hat{\delta}, (Q', \Sigma', \delta', q'_0, F'), (Q_1, Q_2, \Sigma, \delta_1, \delta_2, q^1_0, q^2_0, F_1, F_2)$

**Induction:** $|w| = n + 1$

Split $w = xa$, where $|x| = n$

$\hat{\delta}'(q'_0, w) = \hat{\delta}'(q'_0, xa)$  

For $n = 0$, $|w| = 1$ so $x = \epsilon$  

by definition of $w$

$\hat{\delta}'(q'_0, x) = [p_1, p_2] \iff (\hat{\delta}_1(q^1_0, x) = p_1) \land (\hat{\delta}_2(q^2_0, x) = p_2)$  

by inductive hyp.

$\delta'([p_1, p_2], a) = [r_1, r_2] \iff (\delta_1(p_1, a) = r_1) \land (\delta_2(p_2, a) = r_2)$  

by definition of $\delta'$
Part 2: Proof by Induction

Defined: \( w, \hat{\delta}, (Q', \Sigma', \delta', q'_0, F') \), \( (Q_1, Q_2, \Sigma, \delta_1, \delta_2, q'_0, q^1_0, q^2_0, F_1, F_2) \)

**Induction:** \(|w| = n + 1\)

Split \( w = xa \), where \(|x| = n\)

\( \hat{\delta}'(q'_0, w) = \hat{\delta}'(q'_0, xa) \)

For \( n = 0, |w| = 1 \) so \( x = \epsilon \) \( (2) \)

by definition of \( w \)

\( \hat{\delta}'(q'_0, x) = [p_1, p_2] \iff \left( \hat{\delta}_1(q^1_0, x) = p_1 \right) \land \left( \hat{\delta}_2(q^2_0, x) = p_2 \right) \)

(3) by inductive hyp.

\( \delta'([p_1, p_2], a) = [r_1, r_2] \iff \left( \delta_1(p_1, a) = r_1 \right) \land \left( \delta_2(p_2, a) = r_2 \right) \)

(4) by definition of \( \delta' \)

\( \delta' \left( \hat{\delta}'(q'_0, x), a \right) = [r_1, r_2] \iff \left( \delta_1 \left( \hat{\delta}_1(q^1_0, x), a \right) = r_1 \right) \land \left( \delta_2 \left( \hat{\delta}_2(q^2_0, x), a \right) = r_2 \right) \)

(5) sub. (3) into (4)
Part 2: Proof by Induction

\[ \delta' (\hat{\delta}'(q'_0, x), a) = [r_1, r_2] \iff \left( \delta_1 (\hat{\delta}_1(q'_0, x), a) = r_1 \right) \land \left( \delta_2 (\hat{\delta}_2(q'_0, x), a) = r_2 \right) \]  

(5)  

sub. (3) into (4)
Part 2: Proof by Induction

\[
\delta' \left( \hat{\delta}'(q_0', x), a \right) = [r_1, r_2] \iff \left( \delta_1 \left( \hat{\delta}_1(q_0^1, x), a \right) = r_1 \right) \land \left( \delta_2 \left( \hat{\delta}_2(q_0^2, x), a \right) = r_2 \right) \quad (5)
\]

sub. (3) into (4)

\[
\hat{\delta}'(q_0', xa) = [r_1, r_2] \iff \left( \hat{\delta}_1(q_0^1, xa) = r_1 \right) \land \left( \hat{\delta}_2(q_0^2, xa) = r_2 \right) \quad (6)
\]

by definition of \( \hat{\delta} \)
Part 2: Proof by Induction

\[
\begin{align*}
\delta' \left( \hat{\delta}'(q'_0, x), a \right) &= [r_1, r_2] \iff \\
\left( \delta_1 \left( \hat{\delta}_1(q^1_0, x), a \right) = r_1 \right) \land \left( \delta_2 \left( \hat{\delta}_2(q^2_0, x), a \right) = r_2 \right) 
\end{align*}
\]

(5) sub. (3) into (4)

\[
\hat{\delta}'(q'_0, xa) = [r_1, r_2] \iff \left( \hat{\delta}_1(q^1_0, xa) = r_1 \right) \land \left( \hat{\delta}_2(q^2_0, xa) = r_2 \right) 
\]

(6) by definition of \( \hat{\delta} \)

\[
\hat{\delta}'(q'_0, w) = [r_1, r_2] \iff \left( \hat{\delta}_1(q^1_0, w) = r_1 \right) \land \left( \hat{\delta}_2(q^2_0, w) = r_2 \right) 
\]

(6) by definition of \( w \)

Proved for \(|w| = n + 1\)
Another view: FSA intersection

\[ A_1 \]

\[ A_2 \]

\[ aa^* \cap aaa^* \]

Regular?
DCFSA is just FSA Intersection

Given:

FSA $A_1 = (Q_1, \Sigma, \delta_1, q_0^1, F_1)$
FSA $A_2 = (Q_2, \Sigma, \delta_2, q_0^2, F_2)$
DCFSA $A = (Q_1, Q_2, \Sigma, \delta_1, \delta_2, q_0^1, q_0^2, F_1, F_2)$

Prove:

$L(A) = L(A_1) \cap L(A_2)$
DCFSA is just FSA Intersection

Given:

FSA $A_1 = (Q_1, \Sigma, \delta_1, q_0^1, F_1)$
FSA $A_2 = (Q_2, \Sigma, \delta_2, q_0^2, F_2)$
DCFSA $A = (Q_1, Q_2, \Sigma, \delta_1, \delta_2, q_0^1, q_0^2, F_1, F_2)$

Prove:

$L(A) = L(A_1) \cap L(A_2)$

Prove by dual containment
Direction 1

Direction 1: $L(A) \subseteq L(A_1) \cap L(A_2)$

Let $w \in L(A)$ define \( \hat{\delta}_1(q_0^1, w) \in F_1 \) and \( \hat{\delta}_2(q_0^2, w) \in F_2 \) by definition of $A$ (2)

$w \in L(A_1)$ by definition of $A_1$ (3)

$w \in L(A_2)$ by definition of $A_2$ (4)

$w \in L(A_1) \cap L(A_2)$ by definition of intersection (5)
Direction 2

Direction 2: \( L(A_1) \cap L(A_2) \subseteq L(A) \)

Let \( w \in L(A_1) \cap L(A_2) \) define \( w \in L(A_1) \cap L(A_2) \) by definition of intersection \( w \in L(A_1) \) and \( w \in L(A_2) \) by definition of intersection \( \hat{\delta}_1(q_0^1, w) \in F_1 \) by definition of \( A_1 \) \( \hat{\delta}_2(q_0^2, w) \in F_2 \) by definition of \( A_2 \) \( w \in L(A) \) by definition of \( A \)
We have proven \( L(A) \subseteq L(A_1) \cap L(A_2) \) and \( L(A_1) \cap L(A_2) \subseteq L(A) \).

By dual containment, we can know \( L(A) = L(A_1) \cap L(A_2) \).

As FSA is closed under intersection, we know any DCFSA can be represented in FSA. In other words, DCFSA is regular.
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