11-711: Algorithms for NLP

Recitation #1

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Dual-Control Finite-State Automata

Suppose we have a new type of deterministic FSA called a dual-control FSA (DCFSA). The machine has two transition functions and two sets of states, each of which contains a start state and set of final states. At each step, the machine reads an input symbol and changes state accordingly on each of its controls. The machine accepts an input string if *both* controls are in a final state at the end of the computation. Formally, a DCFSA is defined as

$$A = \langle Q_1, Q_2, \Sigma, \delta_1, \delta_2, q_0^1, q_0^2, F_1, F_2 \rangle$$

The input alphabet Σ is the same for both controls. Q_1, δ_1, q_0^1, and F_1 are the set of states, transition function, initial state, and set of final states for the first control. The transition function δ_1 is a completely defined function, defined on $Q_1 \times \Sigma \to Q_1$. Likewise, Q_2, δ_2, q_0^2, and F_2 are the set of states, transition function, initial state, and set of final states for the second control. The transition function δ_2 is a completely defined function, defined on $Q_2 \times \Sigma \to Q_2$. The language accepted by such a machine is defined as:

$$L(A) = \left\{ \boldsymbol{w} \in \Sigma^* \mid \hat{\delta}_1(q_0^1, \boldsymbol{w}) \in F_1 \text{ and } \hat{\delta}_2(q_0^2, \boldsymbol{w}) \in F_2 \right\}$$

Exercises

1. Show that the set of languages accepted by a DCFSA is regular. (Hint: Construct an equivalent DFA A' that accepts the same language as the DCFSA A. Then show that $\boldsymbol{w} \in L(A)$ if and only if $\boldsymbol{w} \in L(A')$.)

2. Let $A_1 = (Q_1, \Sigma, \delta_1, q_0^1, F_1)$ and $A_2 = (Q_2, \Sigma, \delta_2, q_0^2, F_2)$ be standard DFAs, and let $A = (Q_1, Q_2, \Sigma, \delta_1, \delta_2, q_0^1, q_0^2, F_1, F_2)$ be a DCFSA. Prove that $L(A) = L(A_1) \cap L(A_2)$.

Exercise 1 Solution

The proof that the language of a DCFSA is regular is done in two steps: given a DCFSA, we must construct a deterministic FSA (which we already know accepts a regular language), and then show that the DFA accepts exactly the same language as the DCFSA.

We construct the following DFA $A' = (Q', \Sigma, \delta', q_0', F')$:

- $Q' = \{[q_1, q_2] \mid q_1 \in Q_1, q_2 \in Q_2\}$
- $q_0' = [q_0^1, q_0^2]$
- $F' = \{[f_1, f_2] \mid f_1 \in F_1, f_2 \in F_2\}$
- $\delta'([q_1, q_2], a) = [p_1, p_2]$ if and only if $\delta_1(q_1, \sigma) = p_1$ and $\delta_2(q_2, \sigma) = p_2$, for every $q_1 \in Q_1$, $q_2 \in Q_2$, and $\sigma \in \Sigma$.

We must now show that $\boldsymbol{w} \in L(A)$ if and only if $\boldsymbol{w} \in L(A')$. We do this by induction on the length of $\boldsymbol{w}$, showing that

$$\hat{\delta}'(q_0', \boldsymbol{w}) = [p_1, p_2] \iff \left(\hat{\delta}_1(q_0^1, \boldsymbol{w}) = p_1\right) \wedge \left(\hat{\delta}_2(q_0^2, \boldsymbol{w}) = p_2\right)$$

Base: $|\boldsymbol{w}| = 0$. If $|\boldsymbol{w}| = 0$, then $\boldsymbol{w} = \varepsilon$.

$$\hat{\delta}'(q_0', \varepsilon) = [q_0^1, q_0^2] \iff \left(\hat{\delta}_1(q_0^1, \varepsilon) = q_0^1\right) \wedge \left(\hat{\delta}_2(q_0^2, \varepsilon) = q_0^2\right) \qquad \text{sub. } \varepsilon \text{ into } \hat{\delta} \qquad (1)$$

Induction: $|\boldsymbol{w}| = n + 1$. We split $\boldsymbol{w}$ into $\boldsymbol{w} = \boldsymbol{x}\sigma$, where $|\boldsymbol{x}| = n$.

The string $\boldsymbol{w}$ can be split
$$\hat{\delta}'(q_0', \boldsymbol{w}) = \hat{\delta}'(q_0', \boldsymbol{x}\sigma) \qquad \text{by definition of } \boldsymbol{w} \qquad (2)$$

The inductive hypothesis gives us
$$\hat{\delta}'(q_0', \boldsymbol{x}) = [p_1, p_2] \iff \left(\hat{\delta}_1(q_0^1, \boldsymbol{x}) = p_1\right) \wedge \left(\hat{\delta}_2(q_0^2, \boldsymbol{x}) = p_2\right), \qquad \text{by inductive hyp.} \qquad (3)$$
keeping in mind that we have already proven this for $|\boldsymbol{x}| = 0$ and are about to prove it for any length $\boldsymbol{x}$:

$$\delta'([p_1, p_2], \sigma) = [r_1, r_2] \iff (\delta_1(p_1, \sigma) = r_1) \wedge (\delta_2(p_2, \sigma) = r_2) \qquad \text{by definition of } \delta' \qquad (4)$$

We can apply the equality of $[p_1, p_2]$ from right to left and follow the requirements for p_1 and p_2 to produce
$$\delta'\left(\hat{\delta}'(q_0', \boldsymbol{x}), \sigma\right) = [r_1, r_2] \iff \left(\delta_1\left(\hat{\delta}_1(q_0^1, \boldsymbol{x}), \sigma\right) = r_1\right) \wedge \left(\delta_2\left(\hat{\delta}_2(q_0^2, \boldsymbol{x}), \sigma\right) = r_2\right) \qquad \text{sub. of (3) into (4)} \qquad (5)$$

$$\hat{\delta}'(q_0', \boldsymbol{x}\sigma) = [r_1, r_2] \iff \hat{\delta}_1(q_0^1, \boldsymbol{x}\sigma) = r_1 \wedge \hat{\delta}_2(q_0^2, \boldsymbol{x}\sigma) = r_2 \qquad \text{by definition of } \hat{\delta} \qquad (6)$$

$$\hat{\delta}'(q_0', \boldsymbol{w}) = [r_1, r_2] \iff \left(\hat{\delta}_1(q_0^1, \boldsymbol{w}) = r_1\right) \wedge \left(\hat{\delta}_2(q_0^2, \boldsymbol{w}) = r_2\right) \qquad \text{by definition of } \boldsymbol{w} \qquad (7)$$

The string $\boldsymbol{w}$ is in $L(A)$ if and only if $r_1 \in F_1$ and $r_2 \in F_2$. Based on how we have defined F', this corresponds to A' being in state $[r_1, r_2] \in F'$. Therefore $\boldsymbol{w}$ is in $L(A')$ as well.

This shows that the language of a DCFSA is regular. However, if we want to show that DCFSAs are exactly equivalent in power to DFAs, we must prove that *any* regular language can be captured by a DCFSA. Given a DFA, we must construct an equivalent DCFSA. Informally, this can be done by copying the DFA to both controls of the DCFSA.

Exercise 2 Solution

The proof that the language of a DCFSA A is the intersection of the two languages accepted by its two DFA components A_1 and A_2 is by dual containment. We must show that both $L(A) \subseteq L(A_1) \cap L(A_2)$ and $L(A_1) \cap L(A_2) \subseteq L(A)$.

For the first direction $L(A) \subseteq L(A_1) \cap L(A_2)$:

Let $\boldsymbol{w} \in L(A)$	define	(1)
$\hat{\delta}_1(q_0^1, \boldsymbol{w}) \in F_1$ and $\hat{\delta}_2(q_0^2, \boldsymbol{w}) \in F_2$	by definition of A	(2)
$w \in L(A_1)$	by definition of A_1	(3)
$w \in L(A_2)$	by definition of A_2	(4)
$w \in L(A_1) \cap L(A_2)$	by definition of intersection	(5)

For the second direction $L(A_1) \cap L(A_2) \subseteq L(A)$:

Let $\boldsymbol{w} \in L(A_1) \cap L(A_2)$	define	(6)
$\boldsymbol{w} \in L(A_1)$ and $\boldsymbol{w} \in L(A_2)$	by definition of intersection	(7)
$\hat{\delta}_1(q_0^1, \boldsymbol{w}) \in F_1$	by definition of A_1	(8)
$\hat{\delta}_2(q_0^2, \boldsymbol{w}) \in F_2$	by definition of A_2	(9)
$\boldsymbol{w} \in L(A)$	by definition of A	(10)

Thus the equivalence $L(A) = L(A_1) \cap L(A_2)$ is proved by dual containment.