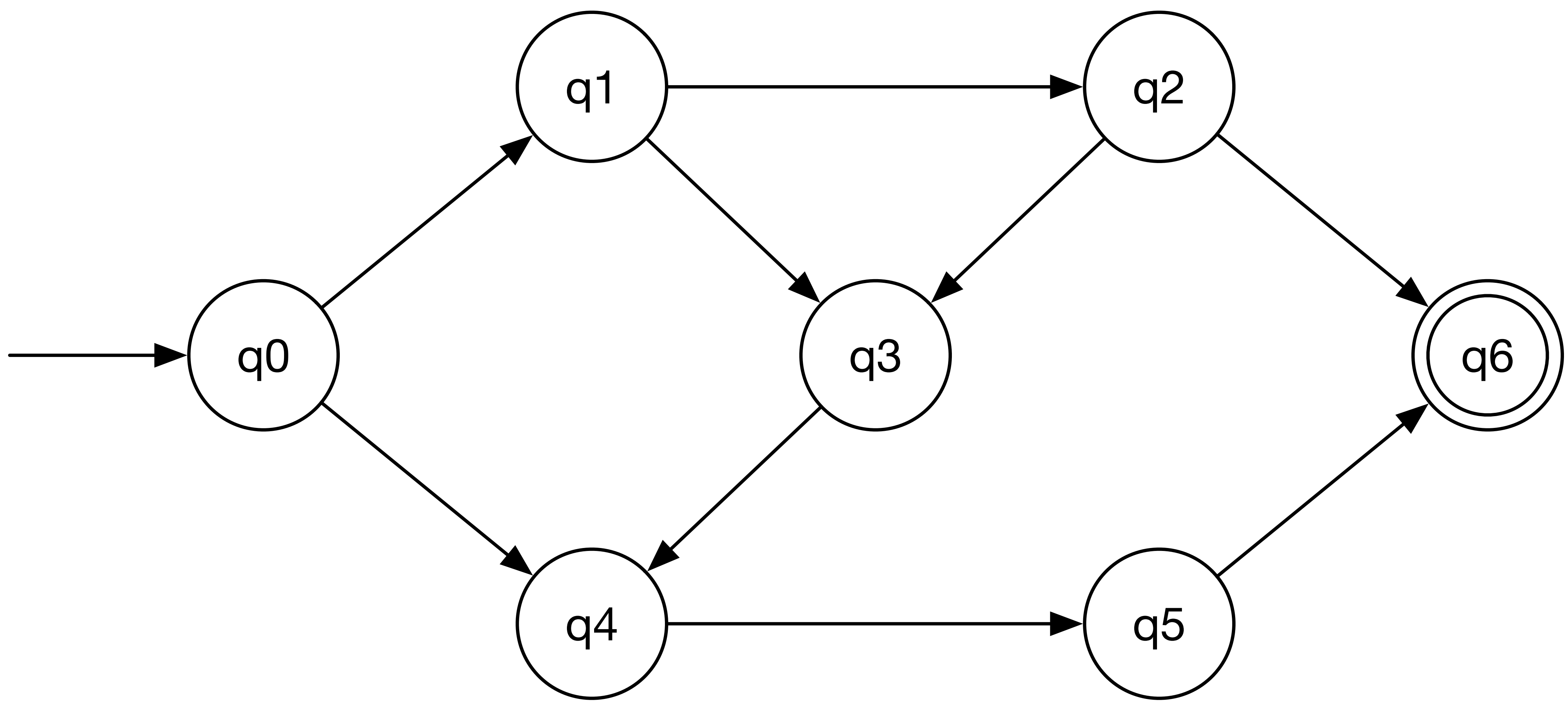
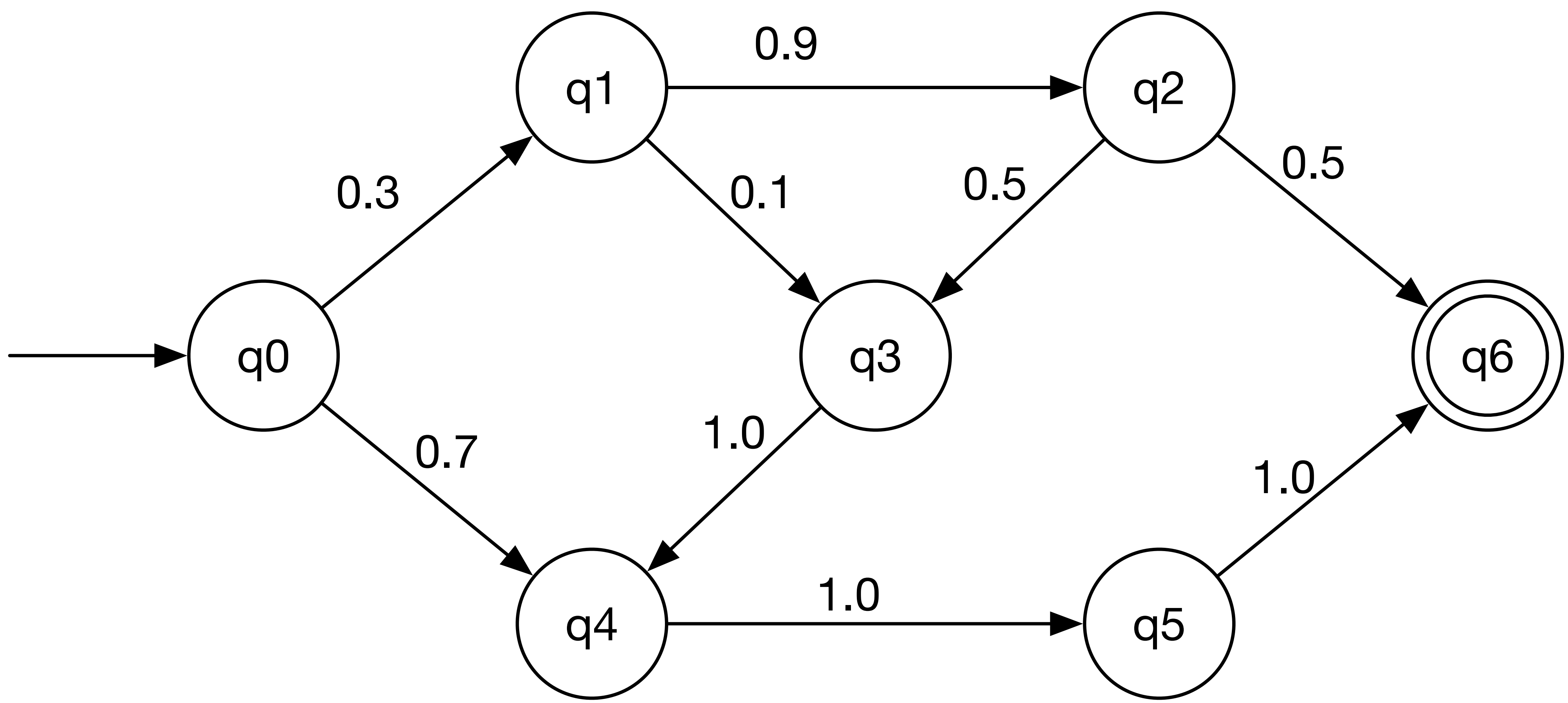
Semiring Examples

11711 Recitation, 2015-10-9

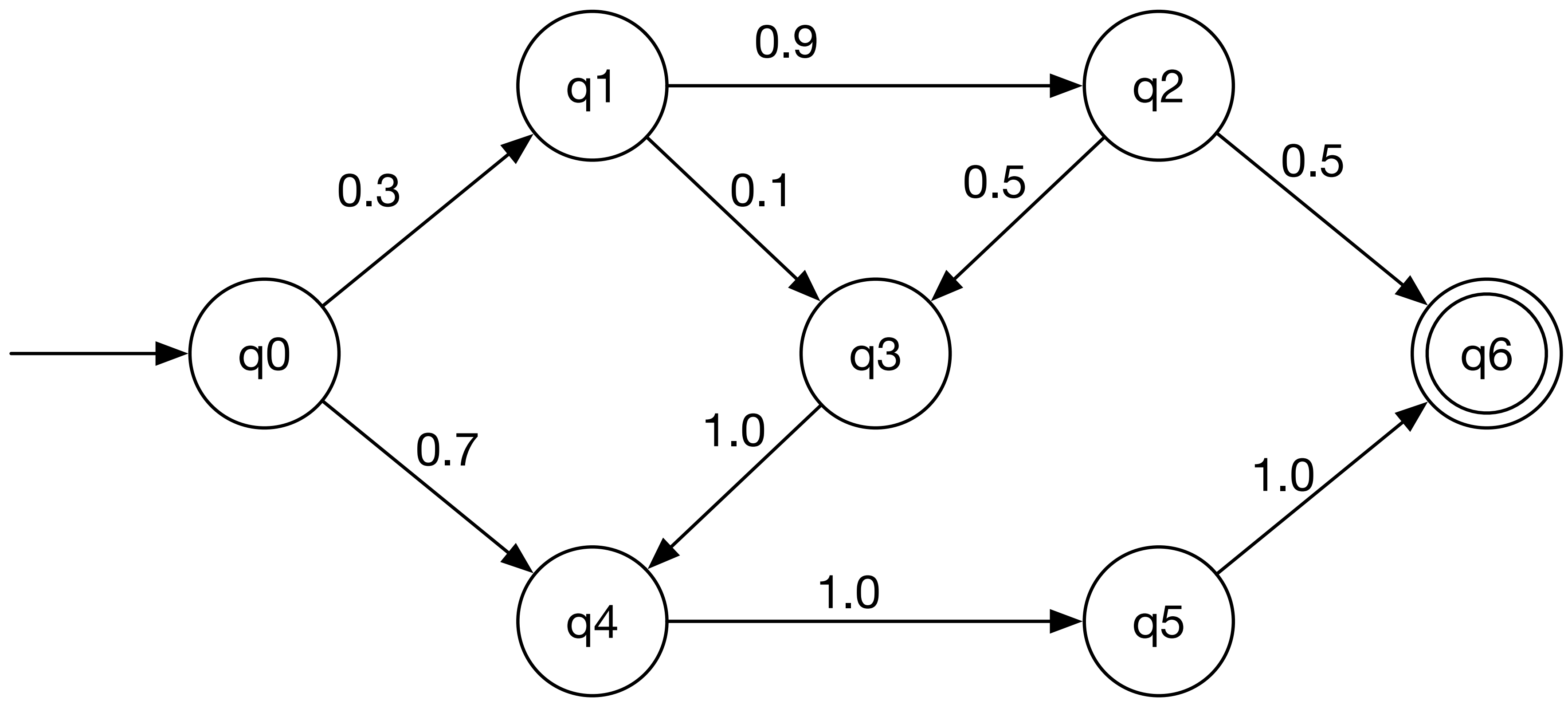
An Example Graph



Probability Semiring



Probability Semiring

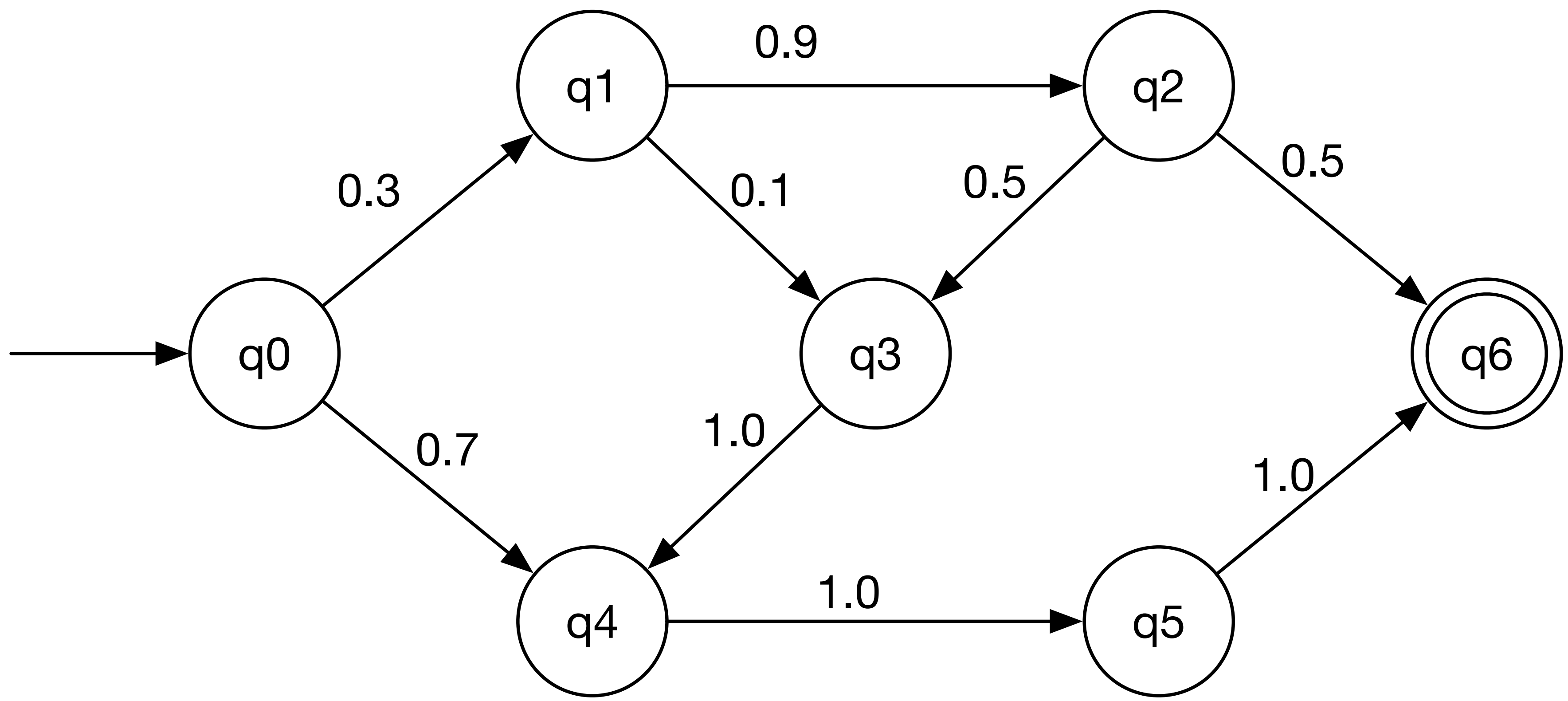


$$Q = \mathbb{R}^+$$
$$\oplus = +$$
$$\otimes = \times$$
$$\bar{0} = 0$$
$$\bar{1} = 1$$

$$\alpha(q_0) = \bar{1} = 1$$

q0	q1	q2	q3	q4	q5	q6
1						

Probability Semiring



$Q = \mathbb{R}^+$

$\oplus = +$

$\otimes = \times$

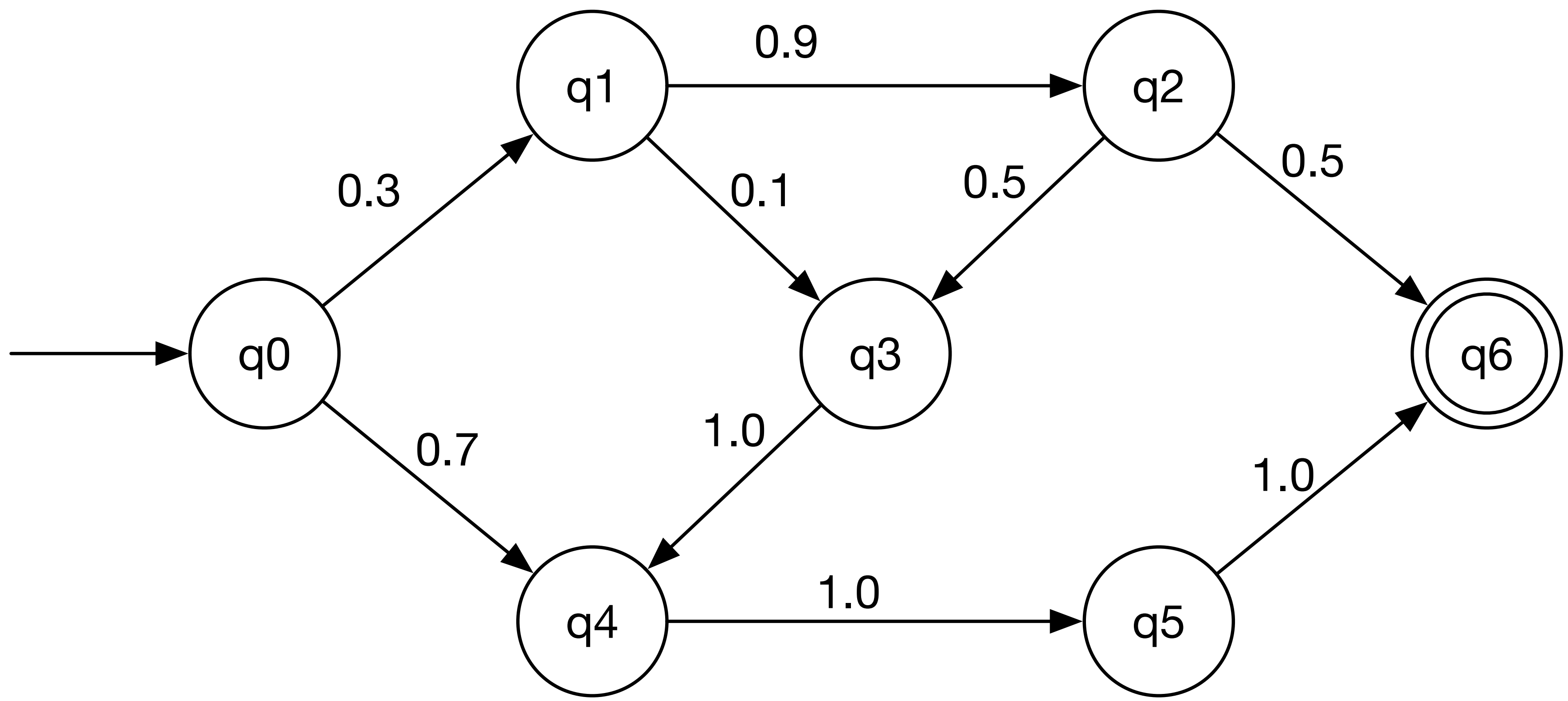
$\bar{0} = 0$

$\bar{1} = 1$

$$\alpha(q_1) = 0.3 \otimes \alpha(q_0) = 0.3 \times 1 = 0.3$$

q0	q1	q2	q3	q4	q5	q6
1	0.3					

Probability Semiring

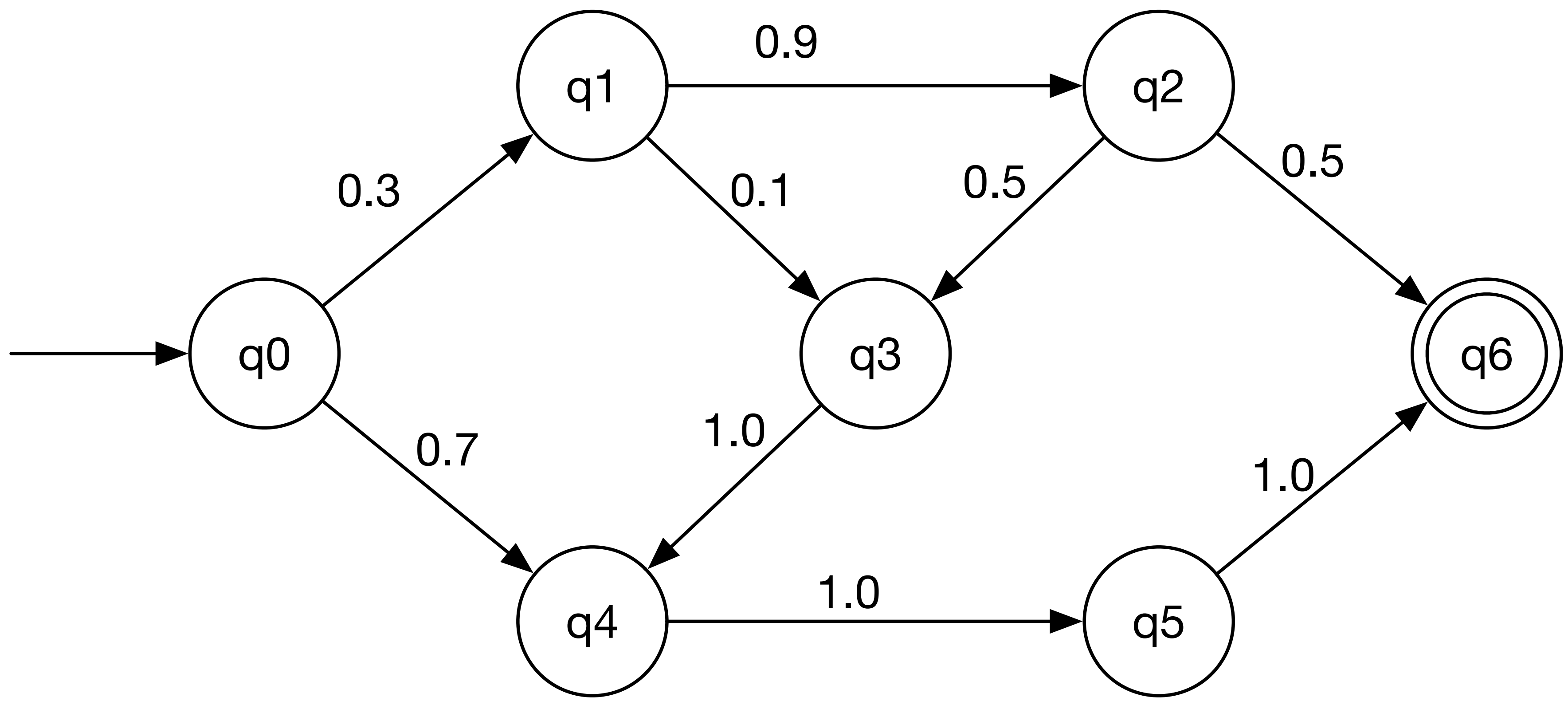


$$Q = \mathbb{R}^+$$
$$\oplus = +$$
$$\otimes = \times$$
$$\bar{0} = 0$$
$$\bar{1} = 1$$

$$\alpha(q_2) = 0.9 \otimes \alpha(q_1) = 0.9 \times 0.3 = 0.27$$

q0	q1	q2	q3	q4	q5	q6
1	0.3	0.27				

Probability Semiring

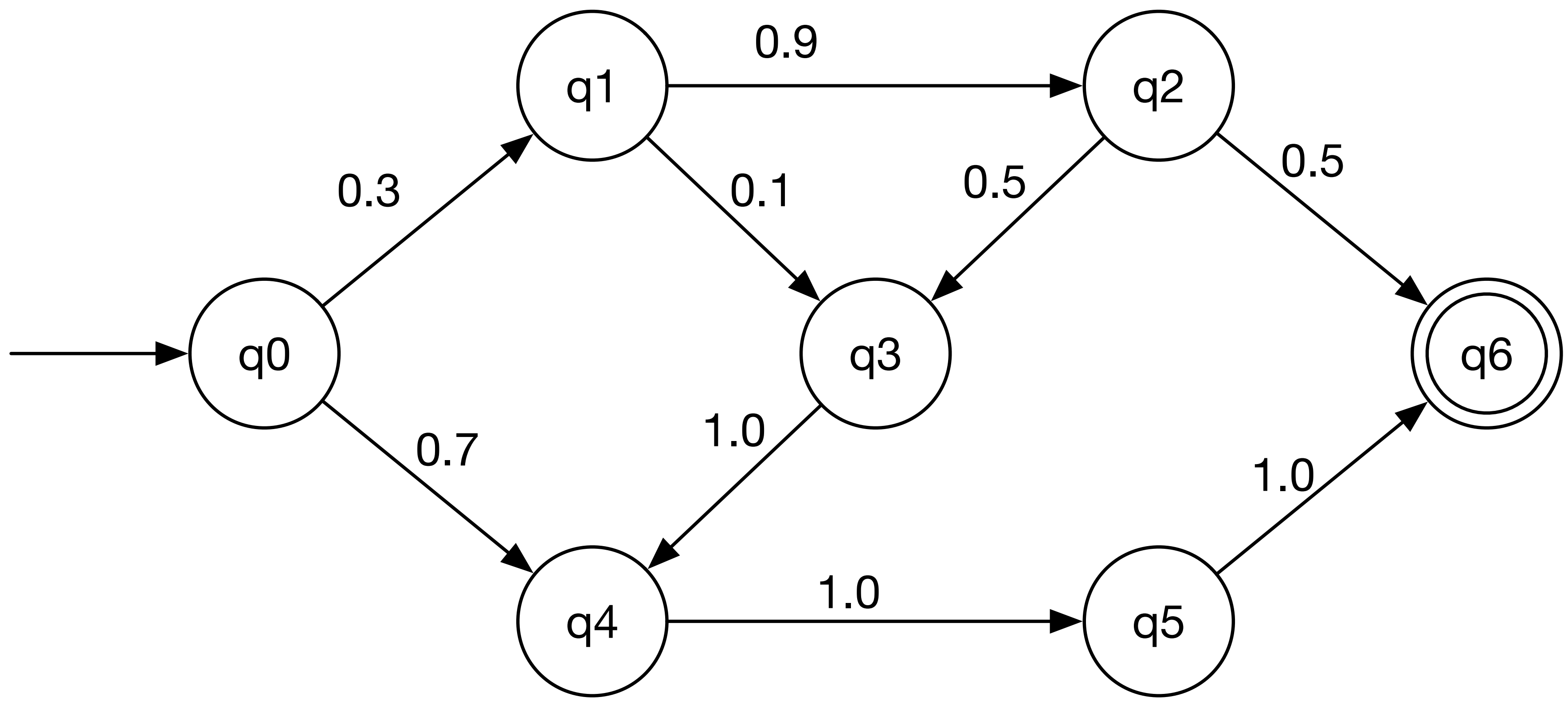


$$Q = \mathbb{R}^+$$
$$\oplus = +$$
$$\otimes = \times$$
$$\bar{0} = 0$$
$$\bar{1} = 1$$

$$\alpha(q_3) = 0.1 \otimes \alpha(q_1) \oplus 0.5 \otimes \alpha(q_2) = 0.1 \times 0.3 + 0.5 \times 0.27 = 0.165$$

q0	q1	q2	q3	q4	q5	q6
1	0.3	0.27	0.165			

Probability Semiring

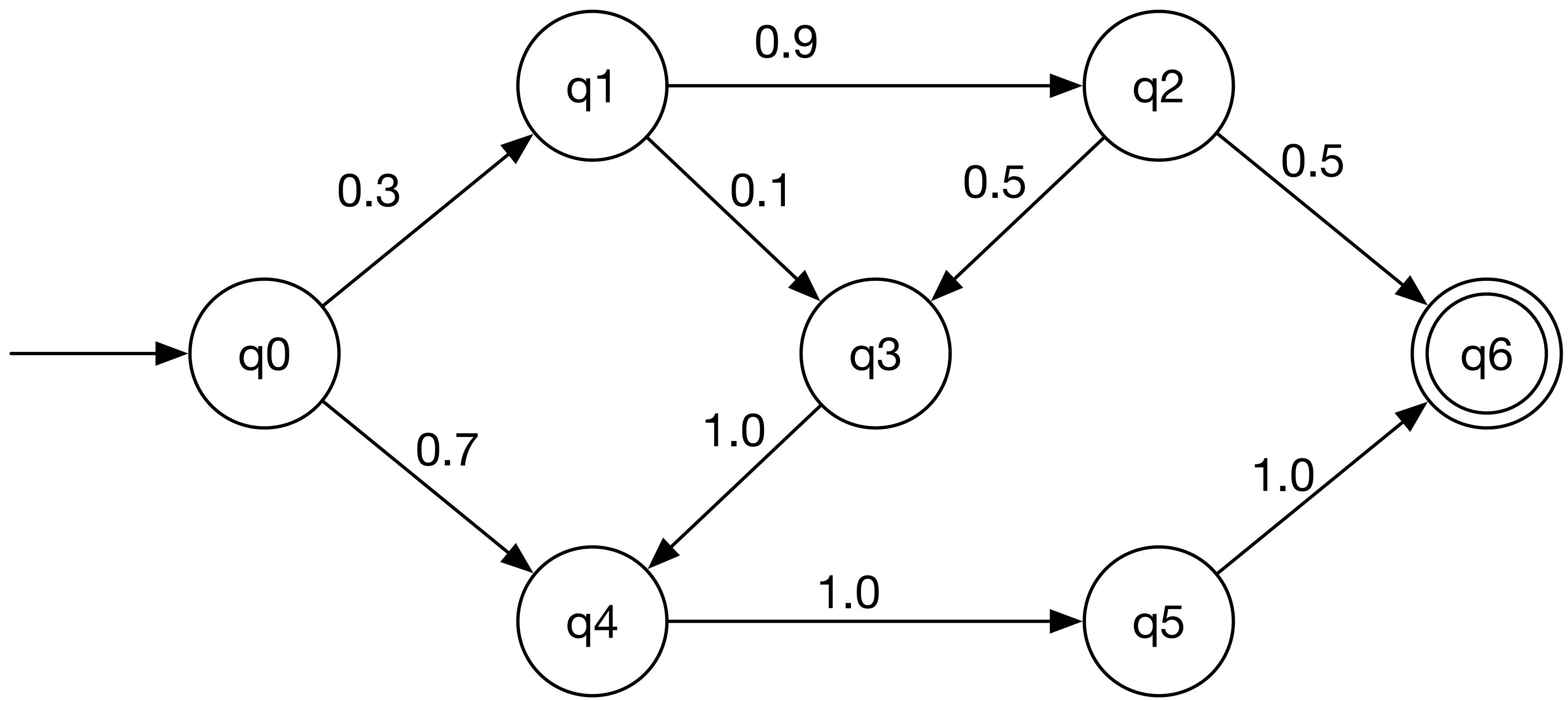


$$Q = \mathbb{R}^+$$
$$\oplus = +$$
$$\otimes = \times$$
$$\bar{0} = 0$$
$$\bar{1} = 1$$

$$\alpha(q_4) = 0.7 \otimes \alpha(q_0) \oplus 1.0 \otimes \alpha(q_3) = 0.7 \times 1.0 + 1.0 \times 0.165 = 0.865$$

q0	q1	q2	q3	q4	q5	q6
1	0.3	0.27	0.165	0.865		

Probability Semiring



$Q = \mathbb{R}^+$

$\oplus = +$

$\otimes = \times$

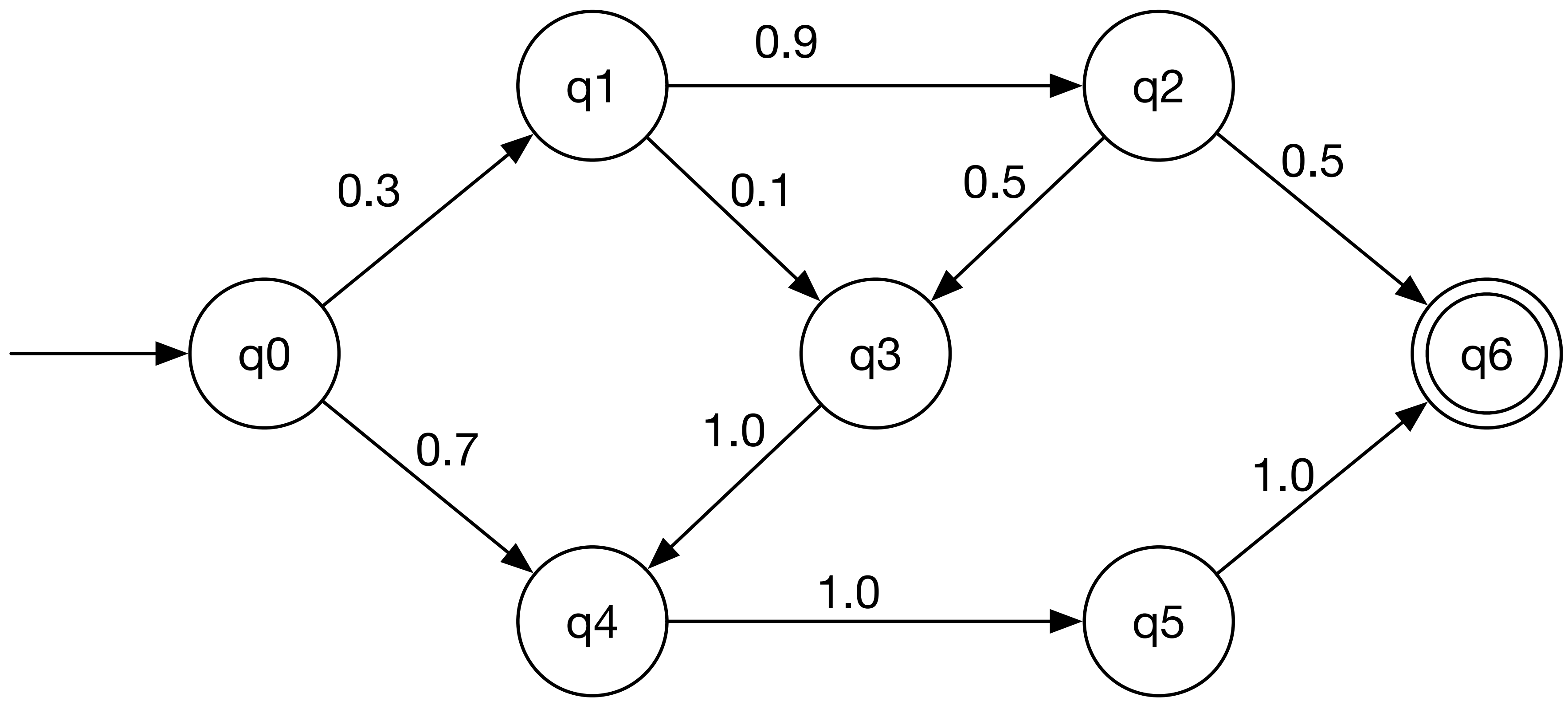
$\bar{0} = 0$

$\bar{1} = 1$

$$\alpha(q_5) = 1.0 \otimes \alpha(q_4) = 1.0 \times 0.865 = 0.865$$

q0	q1	q2	q3	q4	q5	q6
1	0.3	0.27	0.165	0.865	0.865	

Probability Semiring

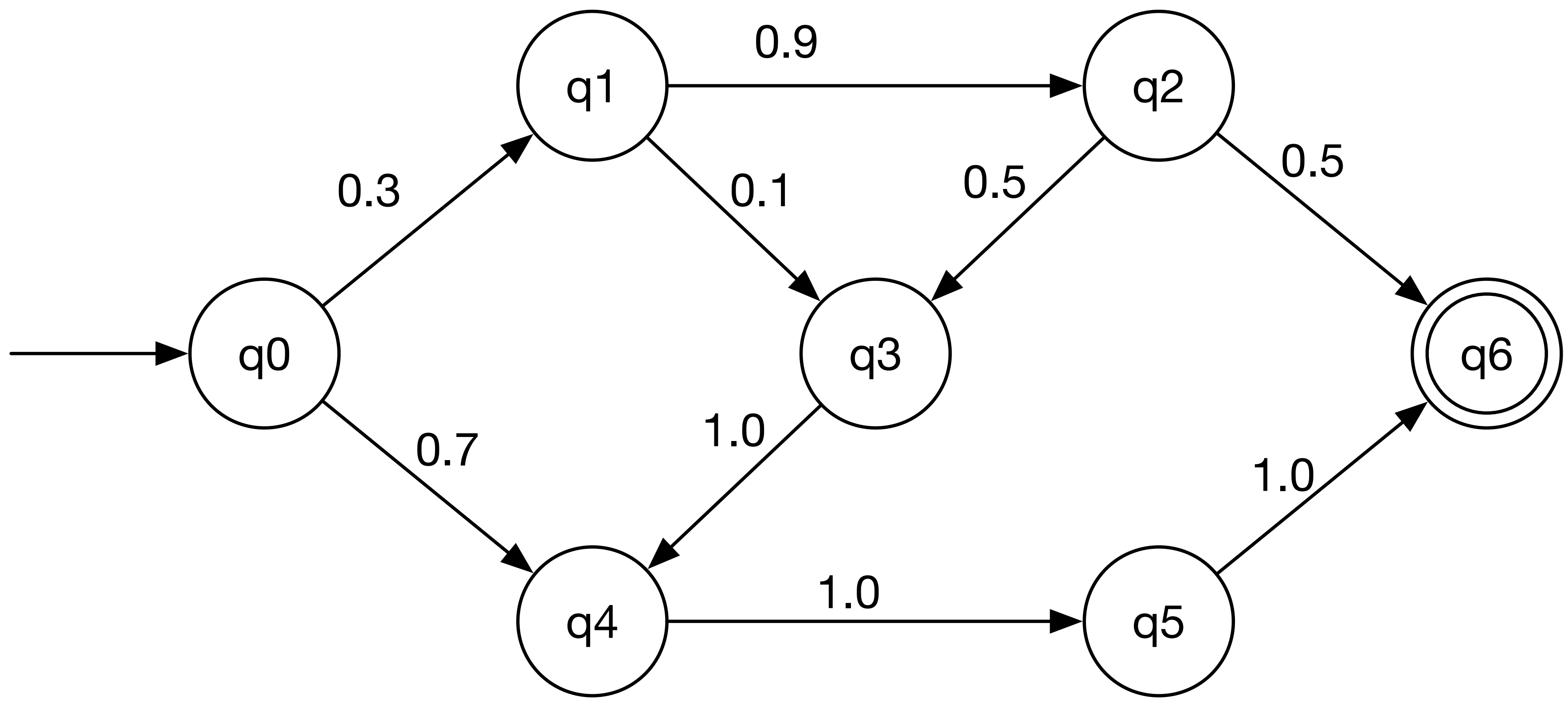


$Q = \mathbb{R}^+$
$\oplus = +$
$\otimes = \times$
$\bar{0} = 0$
$\bar{1} = 1$

$$\alpha(q_6) = 0.5 \otimes \alpha(q_2) \oplus 1.0 \otimes \alpha(q_5) = 0.5 \times 0.27 + 1.0 \times 0.865 = 1.0$$

q0	q1	q2	q3	q4	q5	q6
1	0.3	0.27	0.165	0.865	0.865	1

Viterbi Semiring

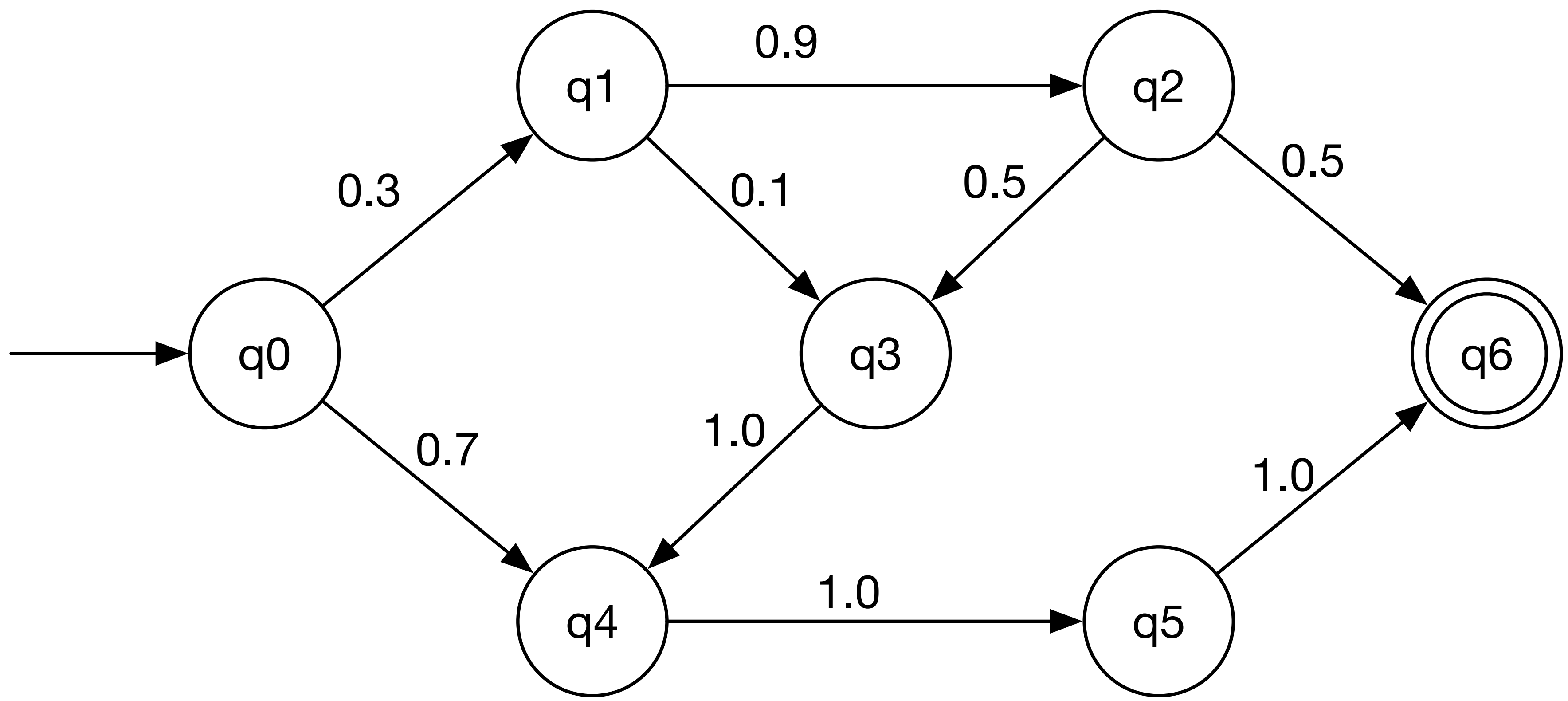


$$Q = \mathbb{R}^+$$
$$\oplus = \max$$
$$\otimes = \times$$
$$\bar{0} = -\infty$$
$$\bar{1} = 1$$

$$\alpha(q_0) = \bar{1} = 1$$

q0	q1	q2	q3	q4	q5	q6
1						

Viterbi Semiring

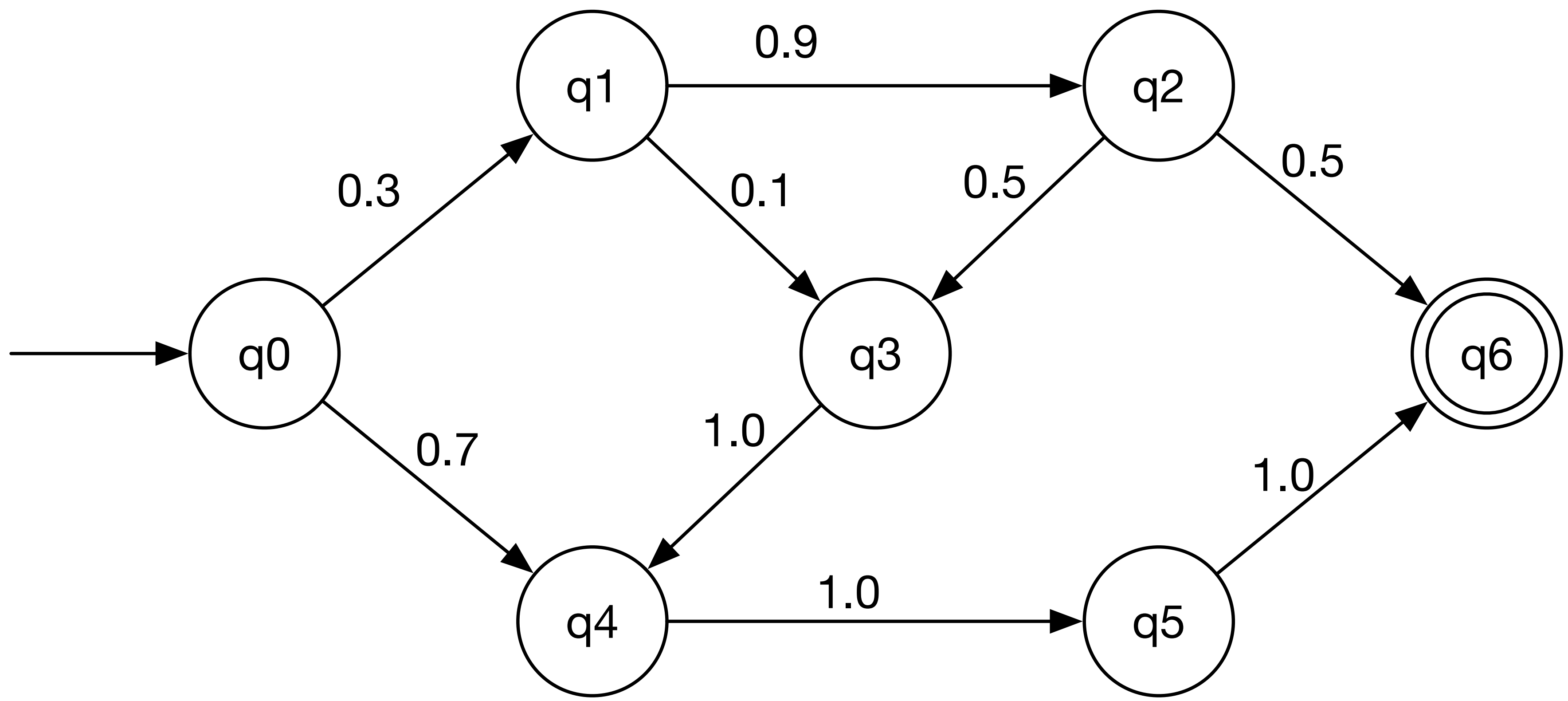


$$Q = \mathbb{R}^+$$
$$\oplus = \max$$
$$\otimes = \times$$
$$\bar{0} = -\infty$$
$$\bar{1} = 1$$

$$\alpha(q_1) = 0.3 \otimes \alpha(q_0) = 0.3 \times 1 = 0.3$$

q0	q1	q2	q3	q4	q5	q6
1	0.3					

Viterbi Semiring



$Q = \mathbb{R}^+$

$\oplus = \max$

$\otimes = \times$

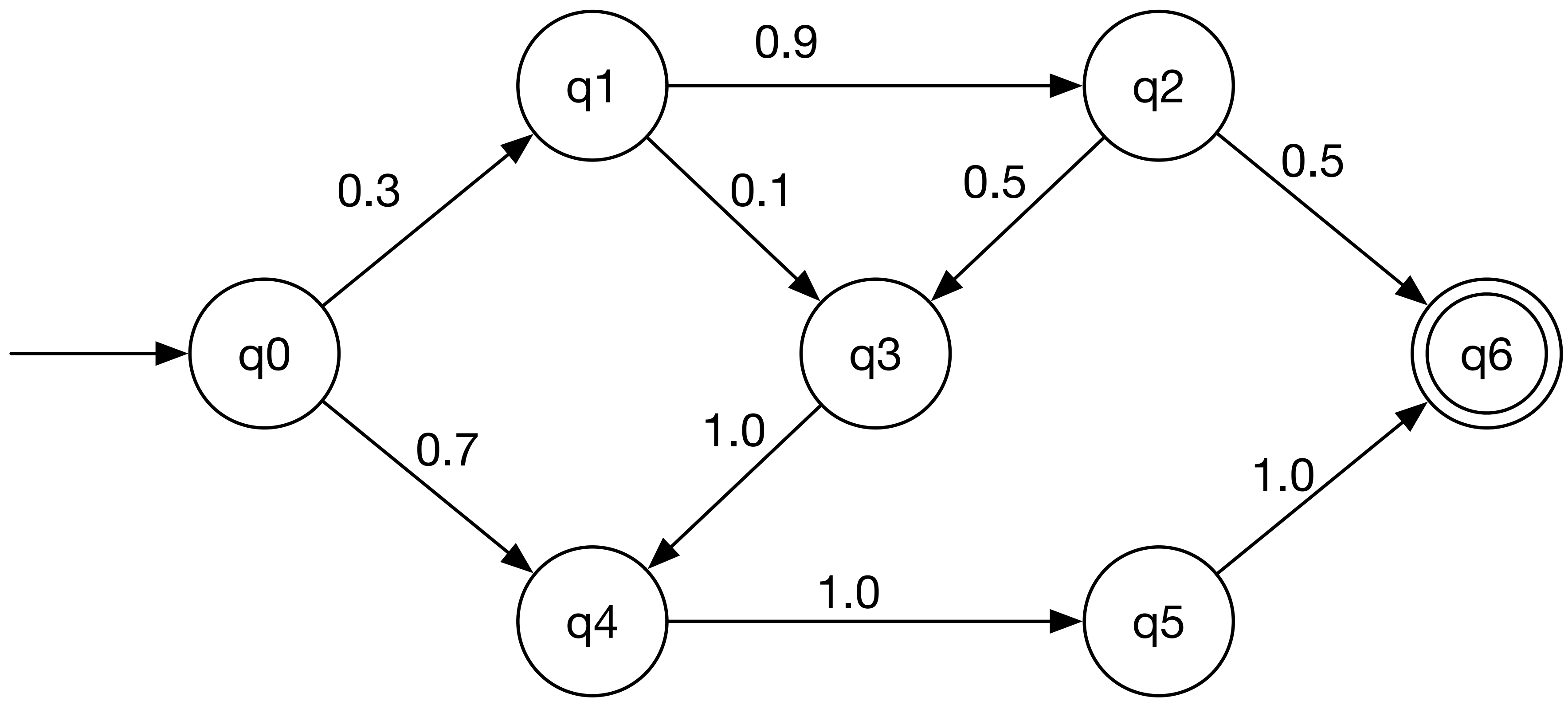
$\bar{0} = -\infty$

$\bar{1} = 1$

$$\alpha(q_2) = 0.9 \otimes \alpha(q_1) = 0.9 \times 0.3 = 0.27$$

q0	q1	q2	q3	q4	q5	q6
1	0.3	0.27				

Viterbi Semiring

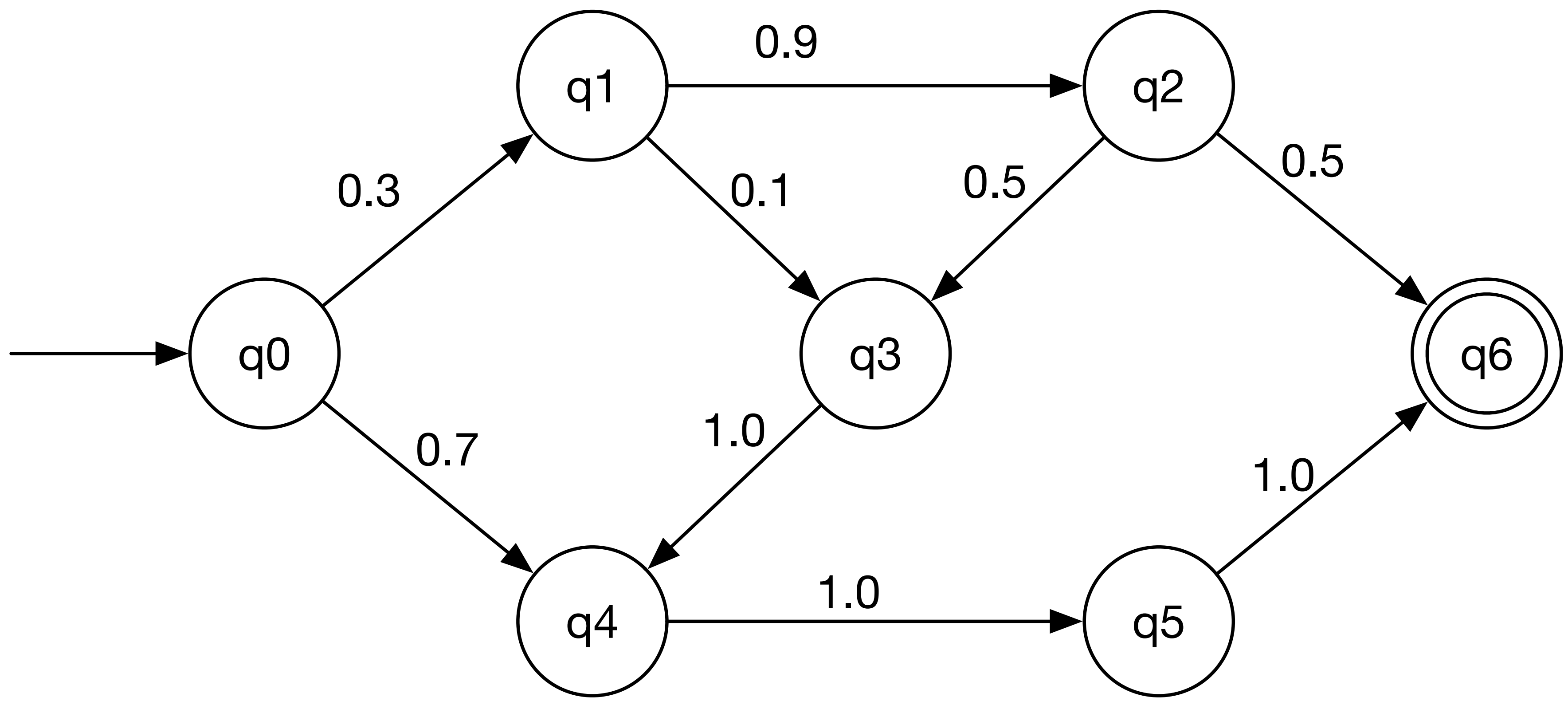


$$Q = \mathbb{R}^+$$
$$\oplus = \max$$
$$\otimes = \times$$
$$\bar{0} = -\infty$$
$$\bar{1} = 1$$

$$\alpha(q_3) = 0.1 \otimes \alpha(q_1) \oplus 0.5 \otimes \alpha(q_2) = \max(0.1 \times 0.3, 0.5 \times 0.27) = 0.135$$

q0	q1	q2	q3	q4	q5	q6
1	0.3	0.27	0.135			

Viterbi Semiring

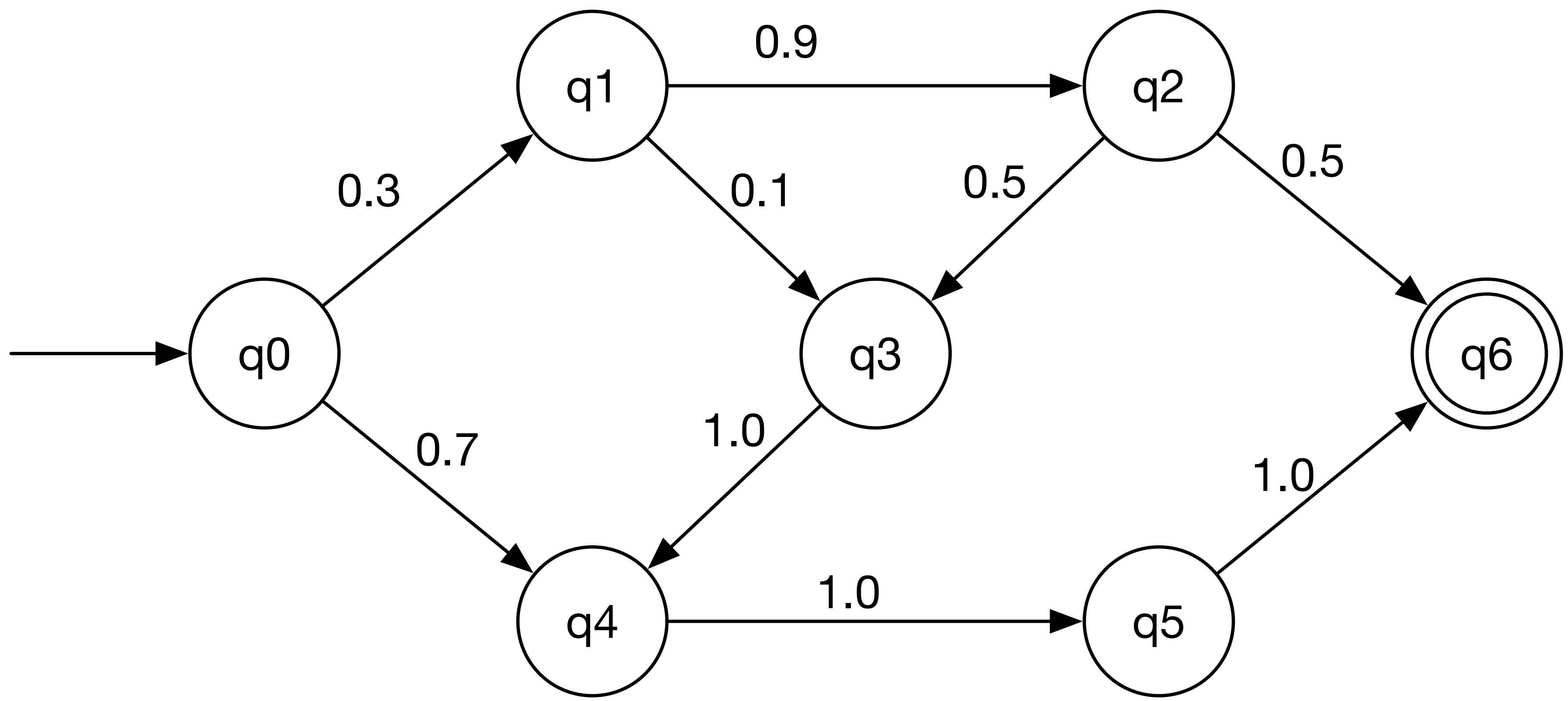


$$Q = \mathbb{R}^+$$
$$\oplus = \max$$
$$\otimes = \times$$
$$\bar{0} = -\infty$$
$$\bar{1} = 1$$

$$\alpha(q_4) = 0.7 \otimes \alpha(q_0) \oplus 1.0 \otimes \alpha(q_3) = \max(0.7 \times 1.0, 1.0 \times 0.135) = 0.7$$

q0	q1	q2	q3	q4	q5	q6
1	0.3	0.27	0.135	0.7		

Viterbi Semiring

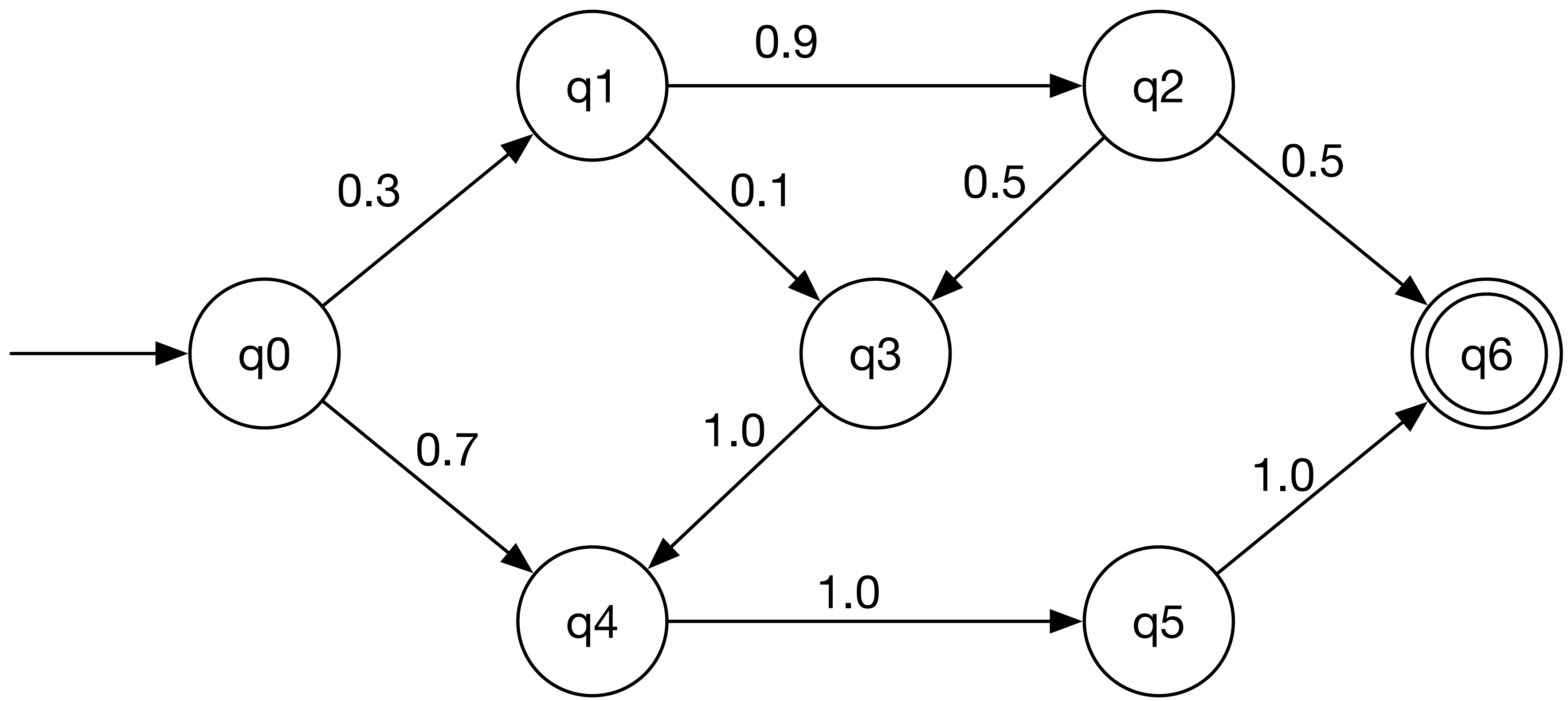


$$Q = \mathbb{R}^+$$
$$\oplus = \max$$
$$\otimes = \times$$
$$\bar{0} = -\infty$$
$$\bar{1} = 1$$

$$\alpha(q_5) = 1.0 \otimes \alpha(q_4) = 1.0 \times 0.7 = 0.7$$

q0	q1	q2	q3	q4	q5	q6
1	0.3	0.27	0.135	0.7	0.7	

Viterbi Semiring

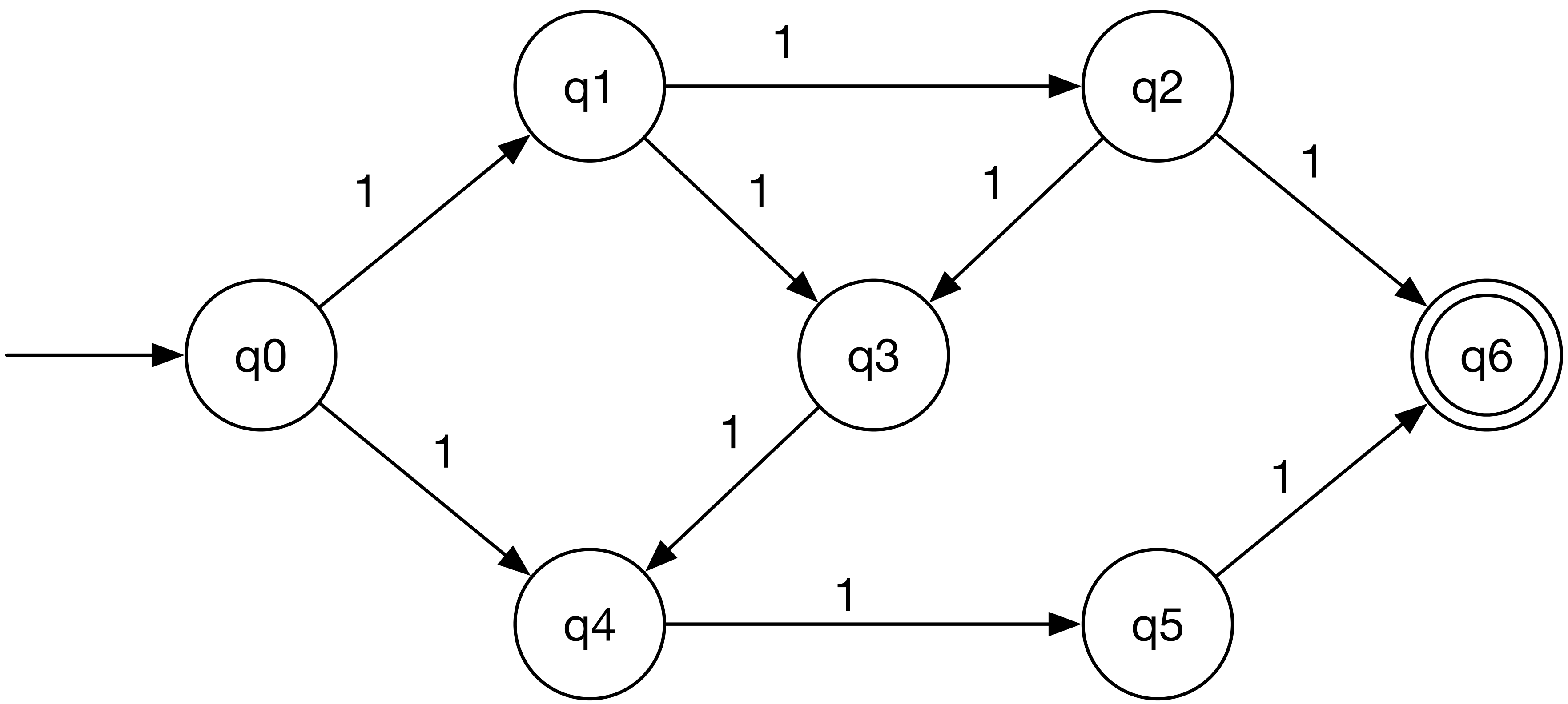


$$Q = \mathbb{R}^+$$
$$\oplus = \max$$
$$\otimes = \times$$
$$\bar{0} = -\infty$$
$$\bar{1} = 1$$

$$\alpha(q_6) = 1.0 \otimes \alpha(q_5) = 1.0 \times 0.7 = 0.7$$

q0	q1	q2	q3	q4	q5	q6
1	0.3	0.27	0.135	0.7	0.7	0.7

Path Counting Semiring

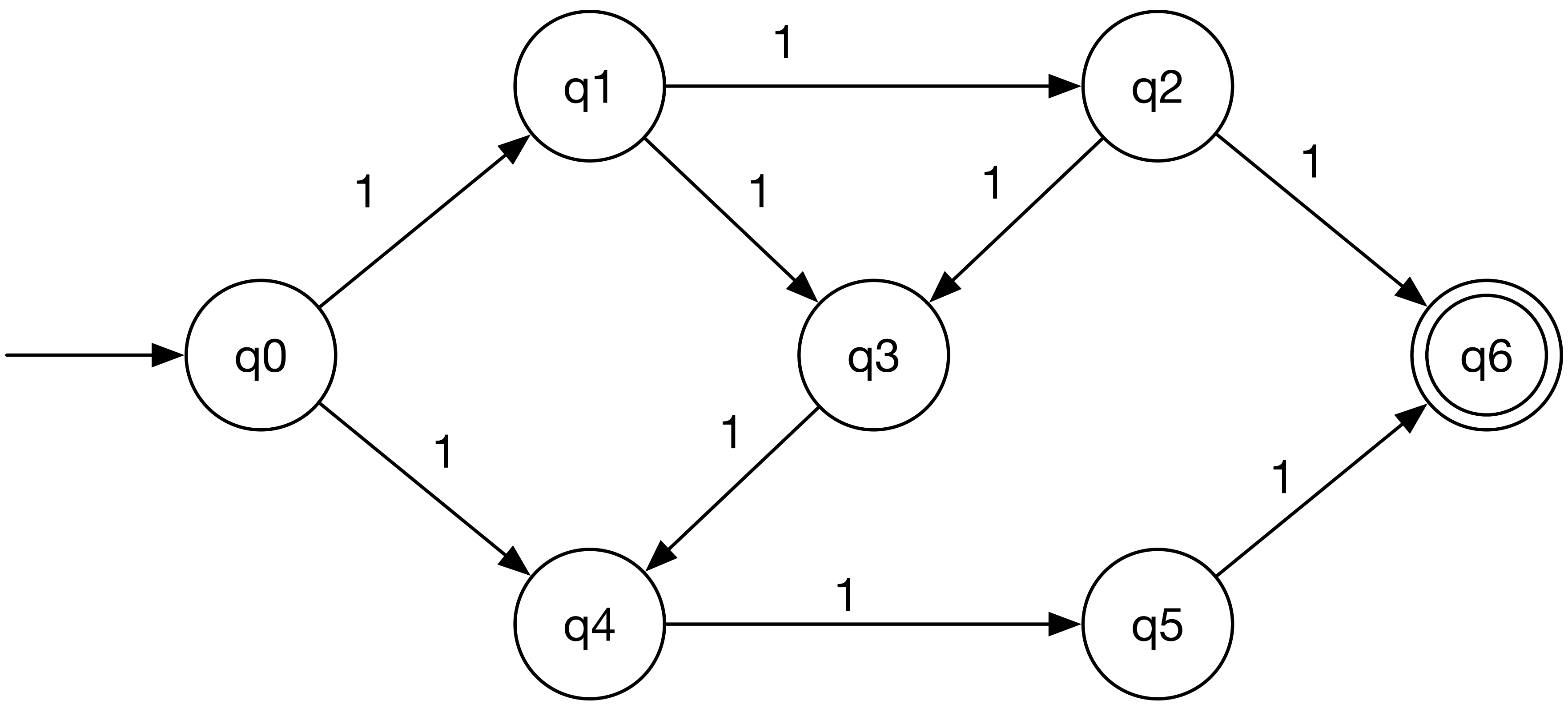


$$Q = \mathbb{Z}^+ \cup \{+\infty\}$$
$$\oplus = +$$
$$\otimes = \times$$
$$\bar{0} = 0$$
$$\bar{1} = 1$$

$$\alpha(q_0) = \bar{1} = 1$$

q0	q1	q2	q3	q4	q5	q6
1						

Path Counting Semiring

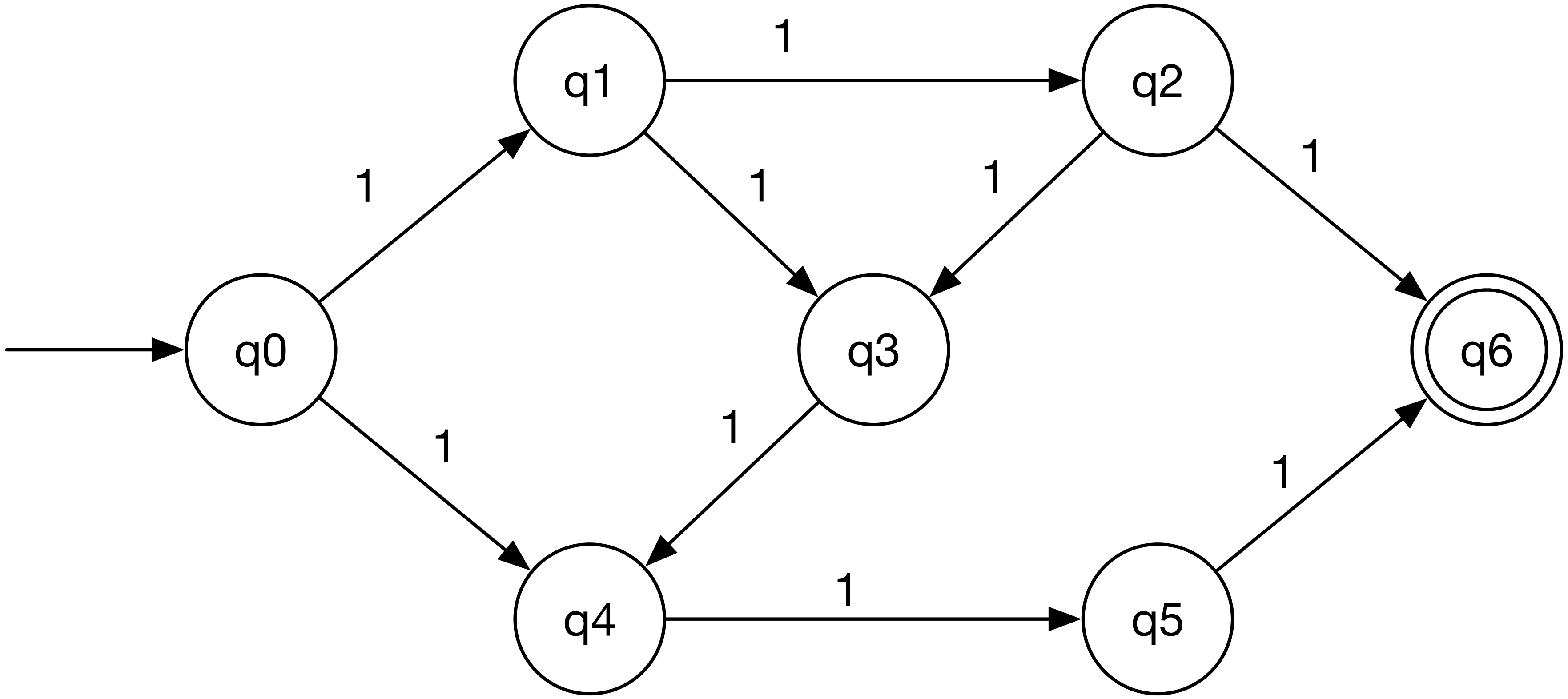


$$Q = \mathbb{Z}^+ \cup \{+\infty\}$$
$$\oplus = +$$
$$\otimes = \times$$
$$\bar{0} = 0$$
$$\bar{1} = 1$$

$$\alpha(q_1) = 1 \otimes \alpha(q_0) = 1 \times 1 = 1$$

q0	q1	q2	q3	q4	q5	q6
1	1					

Path Counting Semiring



$Q = \mathbb{Z}^{+} \cup \{+\infty\}$

$\oplus = +$

$\otimes = \times$

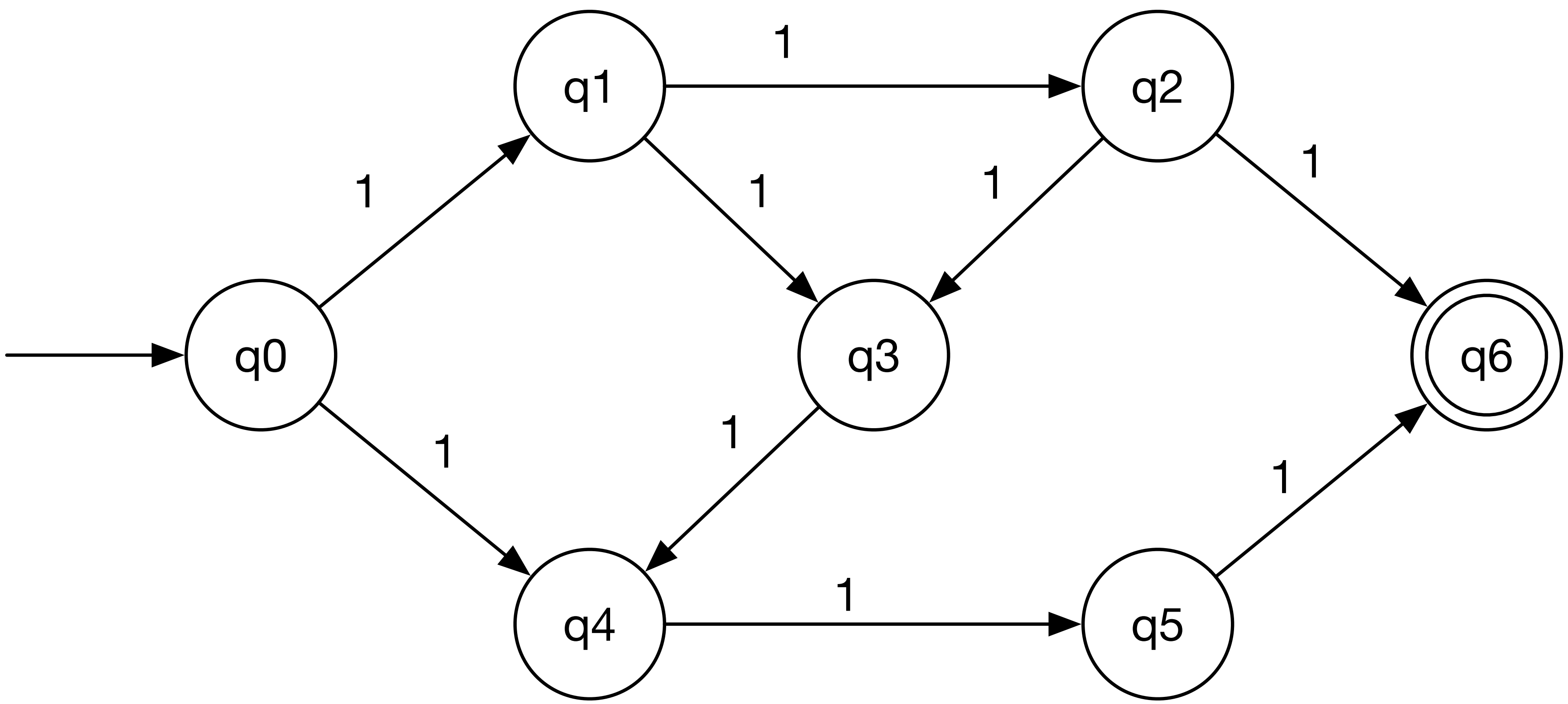
$\bar{0} = 0$

$\bar{1} = 1$

$$\alpha(q_2) = 1 \otimes \alpha(q_1) = 1 \times 1 = 1$$

q0	q1	q2	q3	q4	q5	q6
1	1	1				

Path Counting Semiring

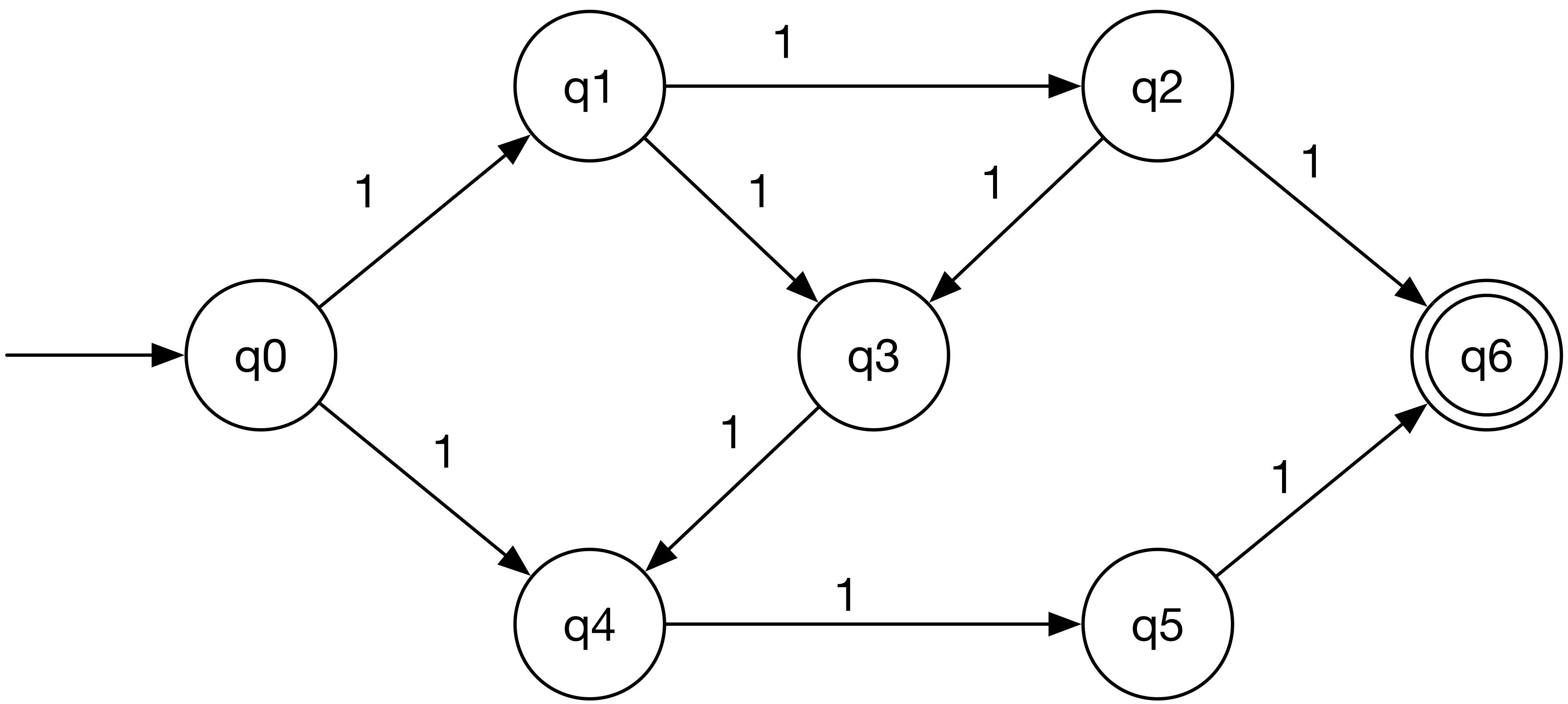


$$Q = \mathbb{Z}^+ \cup \{+\infty\}$$
$$\oplus = +$$
$$\otimes = \times$$
$$\bar{0} = 0$$
$$\bar{1} = 1$$

$$\alpha(q_4) = 1 \otimes \alpha(q_1) \oplus 1 \otimes \alpha(q_3) = 1 \times 1 + 1 \times 2 = 3$$

q0	q1	q2	q3	q4	q5	q6
1	1	1	2	3		

Path Counting Semiring

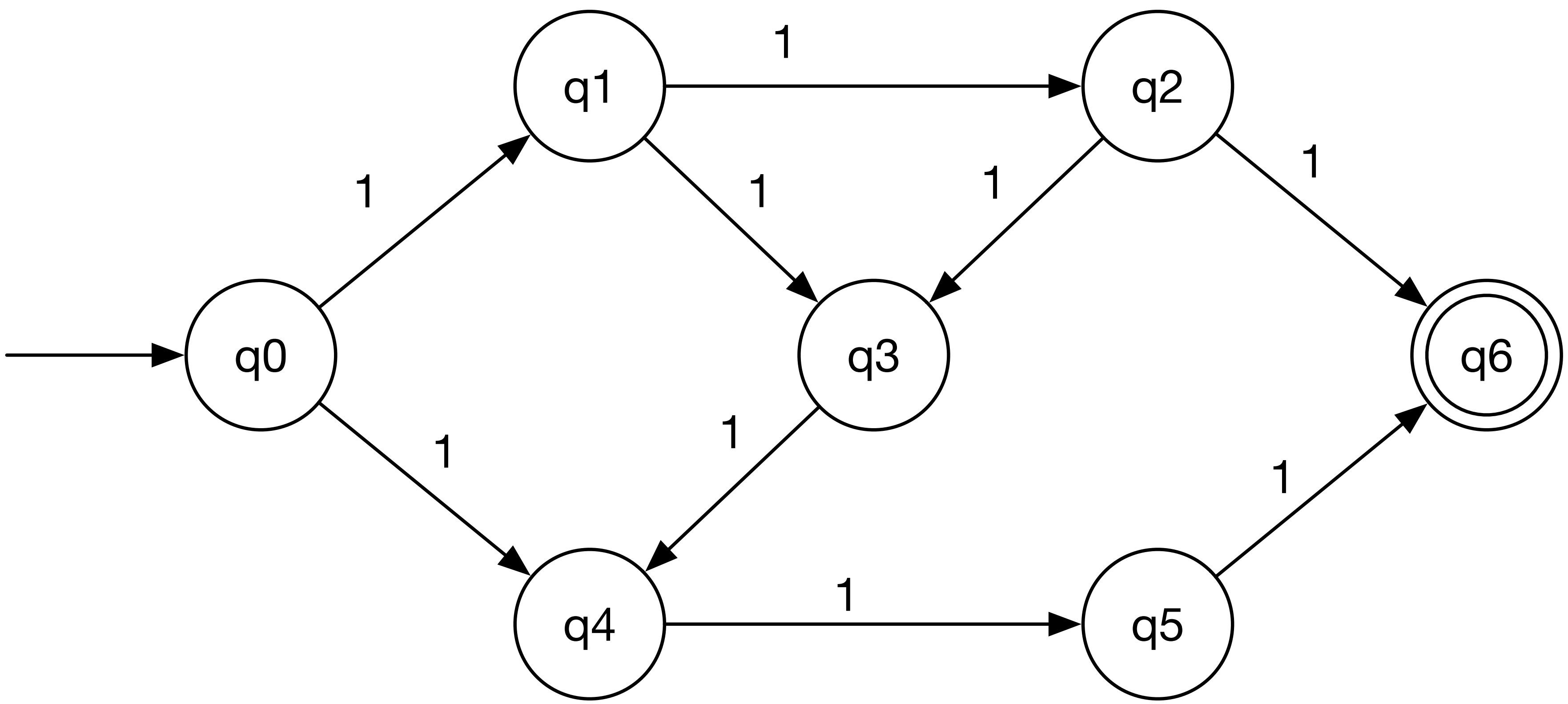


$$Q = \mathbb{Z}^+ \cup \{+\infty\}$$
$$\oplus = +$$
$$\otimes = \times$$
$$\bar{0} = 0$$
$$\bar{1} = 1$$

$$\alpha(q_5) = 1 \otimes \alpha(q_4) = 1 \times 3 = 3$$

q0	q1	q2	q3	q4	q5	q6
1	1	1	2	3	3	

Path Counting Semiring

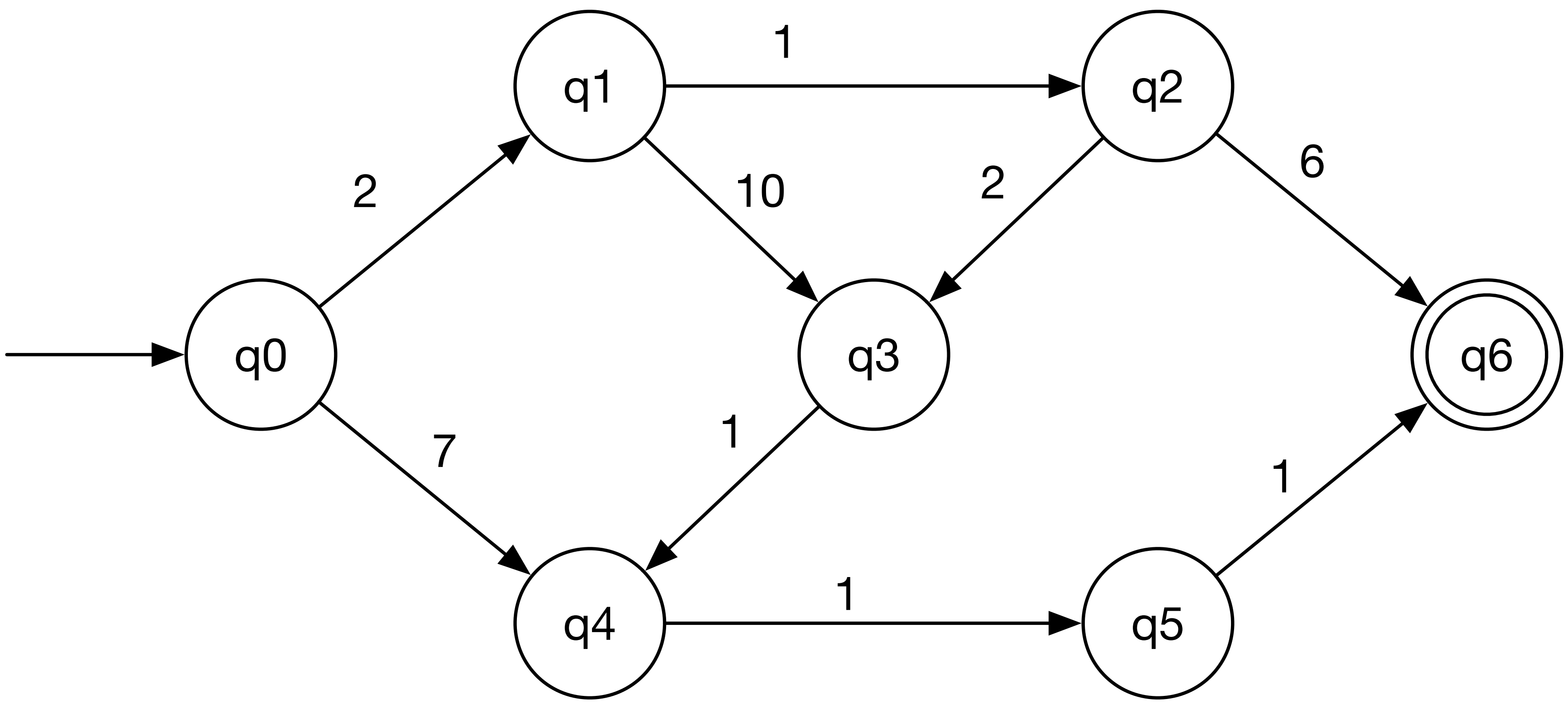


$$Q = \mathbb{Z}^+ \cup \{+\infty\}$$
$$\oplus = +$$
$$\otimes = \times$$
$$\bar{0} = 0$$
$$\bar{1} = 1$$

$$\alpha(q_6) = 1 \otimes \alpha(q_2) \oplus 1 \otimes \alpha(q_5) = 1 \times 1 + 1 \times 3 = 4$$

q0	q1	q2	q3	q4	q5	q6
1	1	1	2	3	3	4

Tropical Semiring

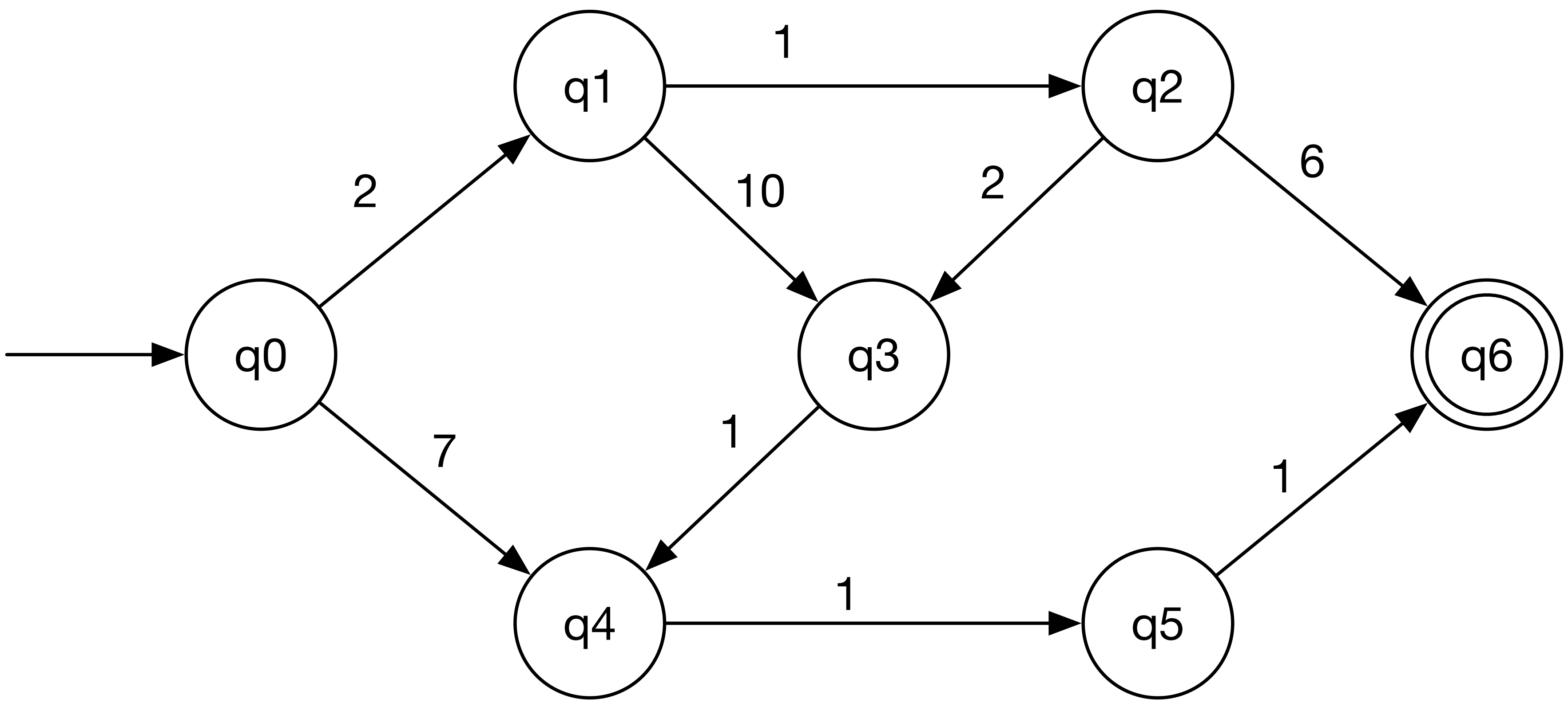


$$Q = \mathbb{R}^+ \cup \{+\infty\}$$
$$\oplus = \min$$
$$\otimes = +$$
$$\bar{0} = +\infty$$
$$\bar{1} = 0$$

$$\alpha(q_0) = \bar{1} = 0$$

q0	q1	q2	q3	q4	q5	q6
0						

Tropical Semiring

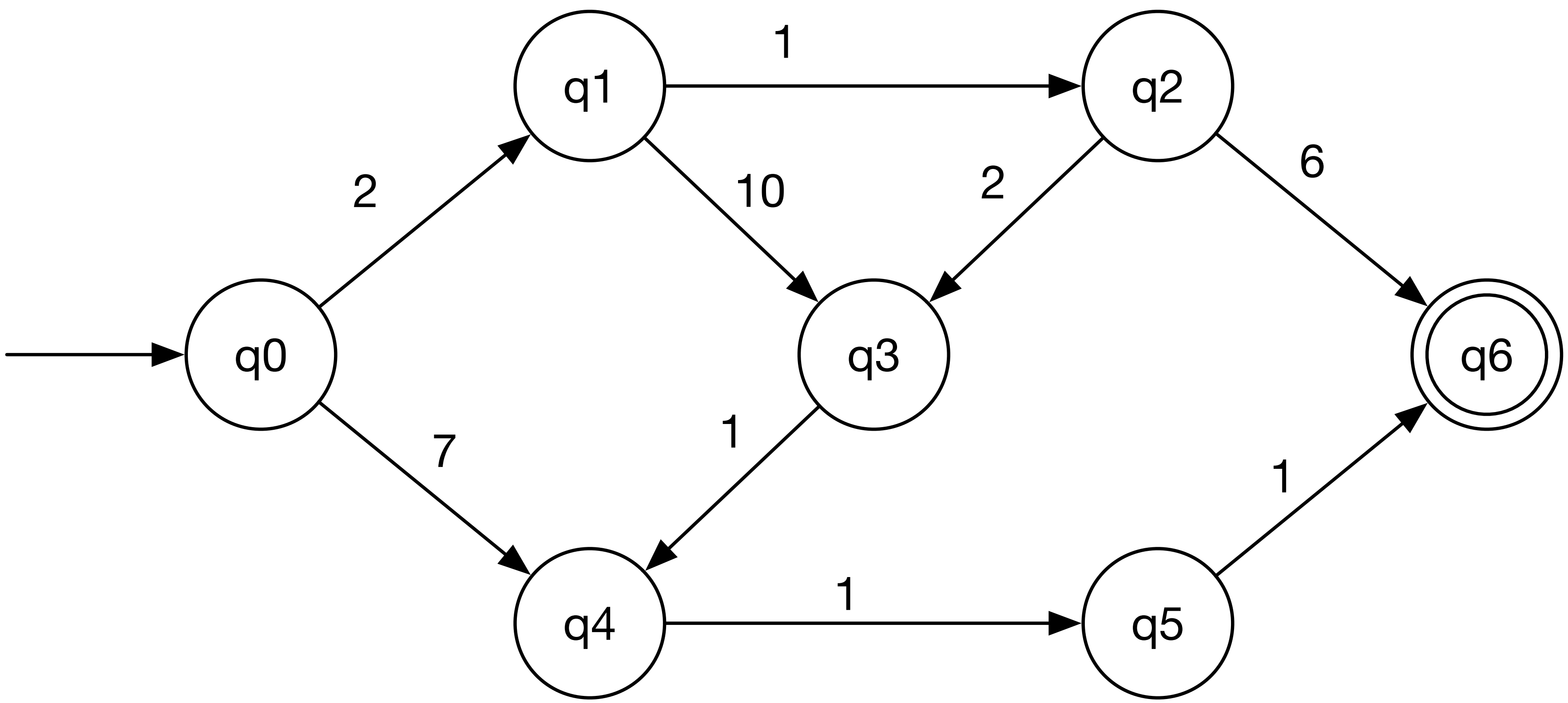


$$Q = \mathbb{R}^+ \cup \{+\infty\}$$
$$\oplus = \min$$
$$\otimes = +$$
$$\bar{0} = +\infty$$
$$\bar{1} = 0$$

$$\alpha(q_2) = 1 \otimes \alpha(q_1) = 1 + 2 = 3$$

q0	q1	q2	q3	q4	q5	q6
0	2	3				

Tropical Semiring

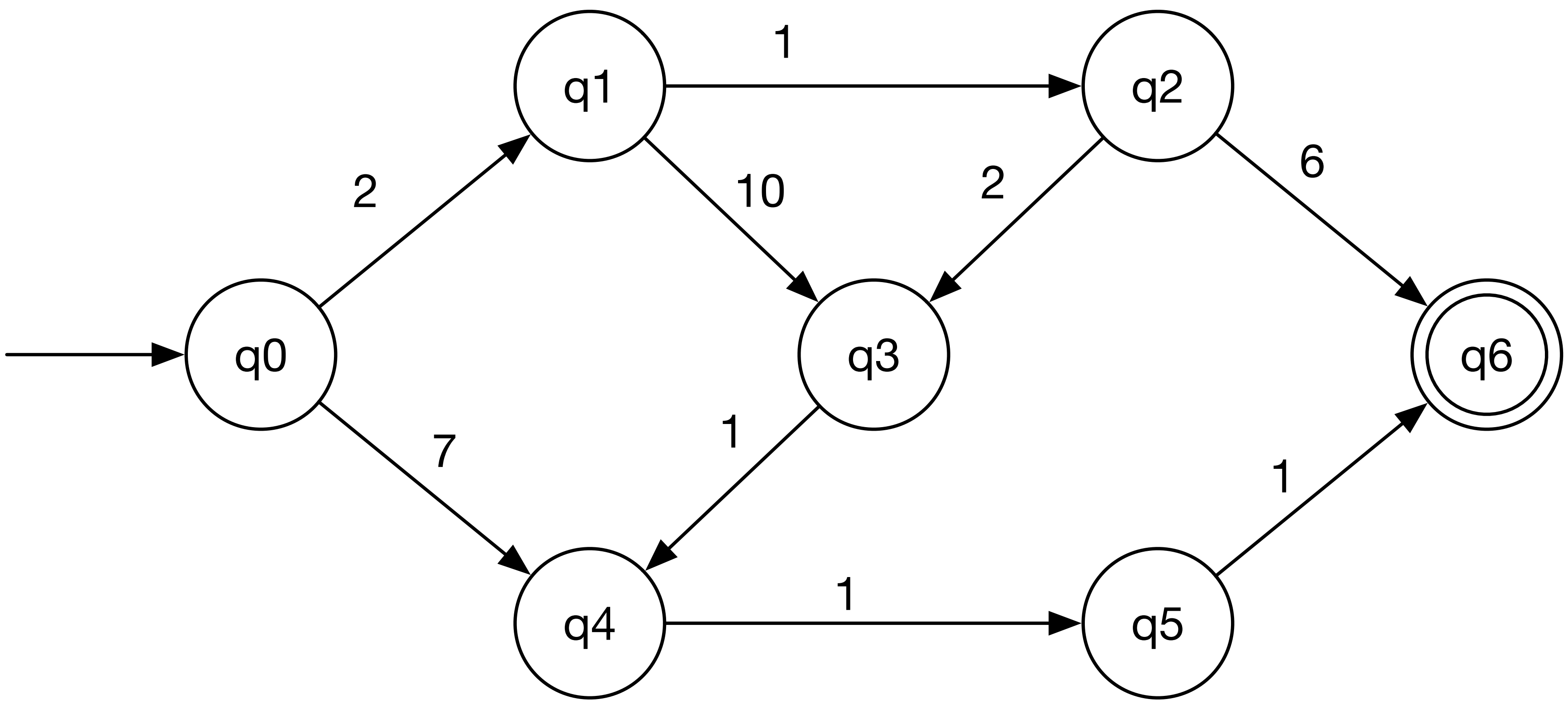


$$Q = \mathbb{R}^+ \cup \{+\infty\}$$
$$\oplus = \min$$
$$\otimes = +$$
$$\bar{0} = +\infty$$
$$\bar{1} = 0$$

$$\alpha(q_3) = 10 \otimes \alpha(q_1) \oplus 2 \otimes \alpha(q_2) = \min(10 + 2, 2 + 3) = 5$$

q0	q1	q2	q3	q4	q5	q6
0	2	3	5			

Tropical Semiring

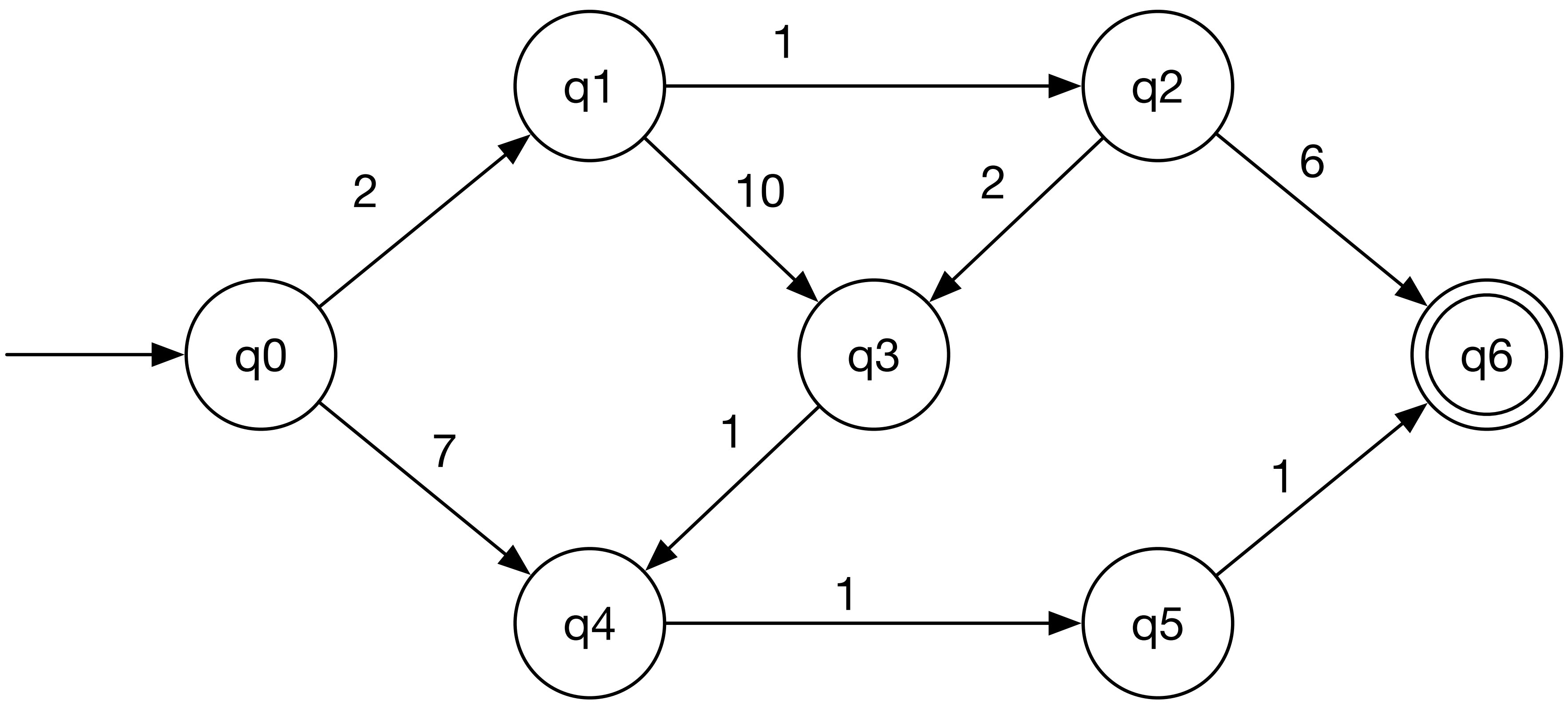


$$Q = \mathbb{R}^+ \cup \{+\infty\}$$
$$\oplus = \min$$
$$\otimes = +$$
$$\bar{0} = +\infty$$
$$\bar{1} = 0$$

$$\alpha(q_4) = 7 \otimes \alpha(q_0) \oplus 1 \otimes \alpha(q_3) = \min(7+0, 1+5) = 6$$

q0	q1	q2	q3	q4	q5	q6
0	2	3	5	6		

Tropical Semiring

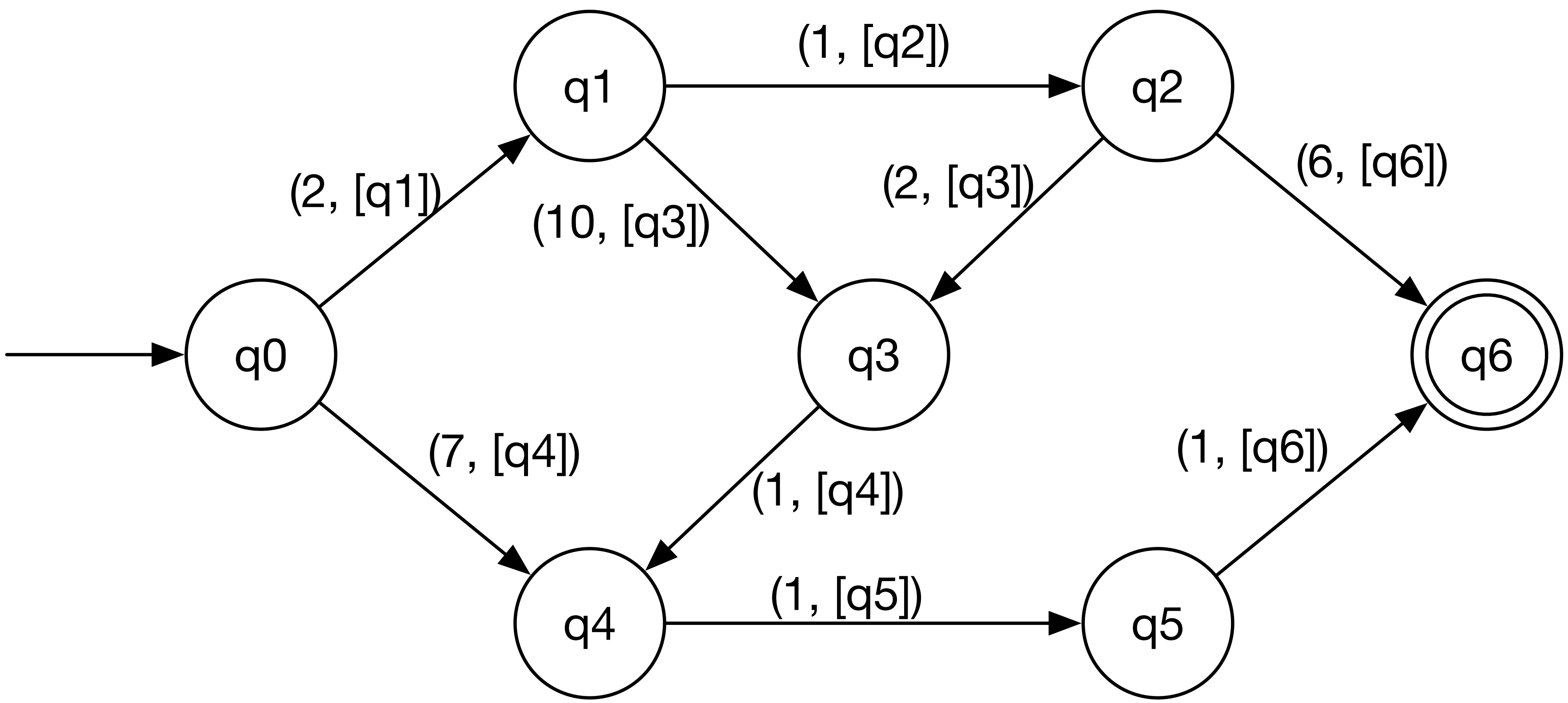


$$Q = \mathbb{R}^+ \cup \{+\infty\}$$
$$\oplus = \min$$
$$\otimes = +$$
$$\bar{0} = +\infty$$
$$\bar{1} = 0$$

$$\alpha(q_6) = 1 \otimes \alpha(q_5) = 1 + 7 = 8$$

q0	q1	q2	q3	q4	q5	q6
0	2	3	5	6	7	8

Best Path Semiring

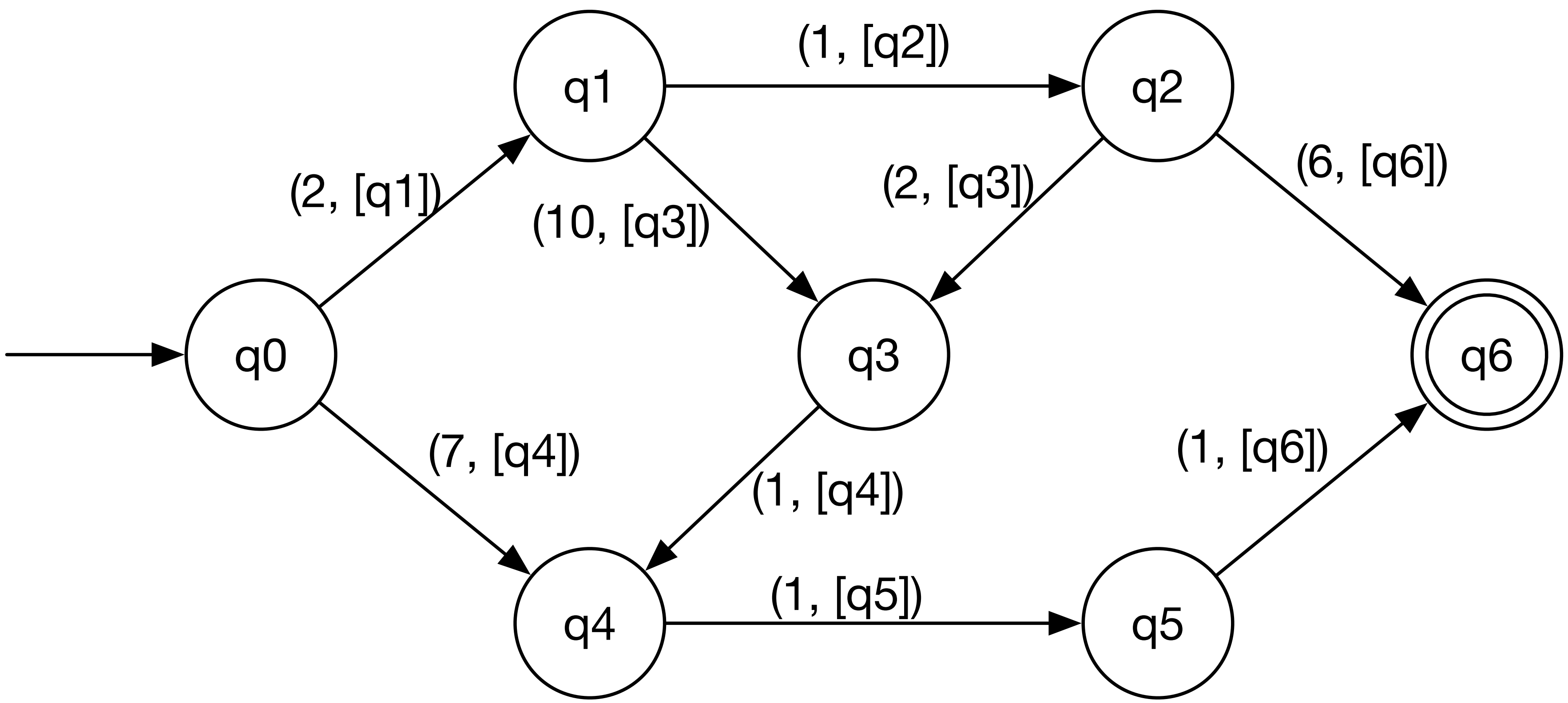


$$Q = (s, p)$$
$$s \in \mathbb{R} \cup \{+\infty\}$$
$$p \in \text{list of states}$$
$$\oplus = \min$$
$$\otimes = (+, \text{append})$$
$$\bar{0} = (+\infty, [])$$
$$\bar{1} = (0, [q_0])$$

$$\alpha(q_0) = \bar{1} = (0, [q_0])$$

q0	q1	q2	q3	q4	q5	q6
(0, [q0])						

Best Path Semiring

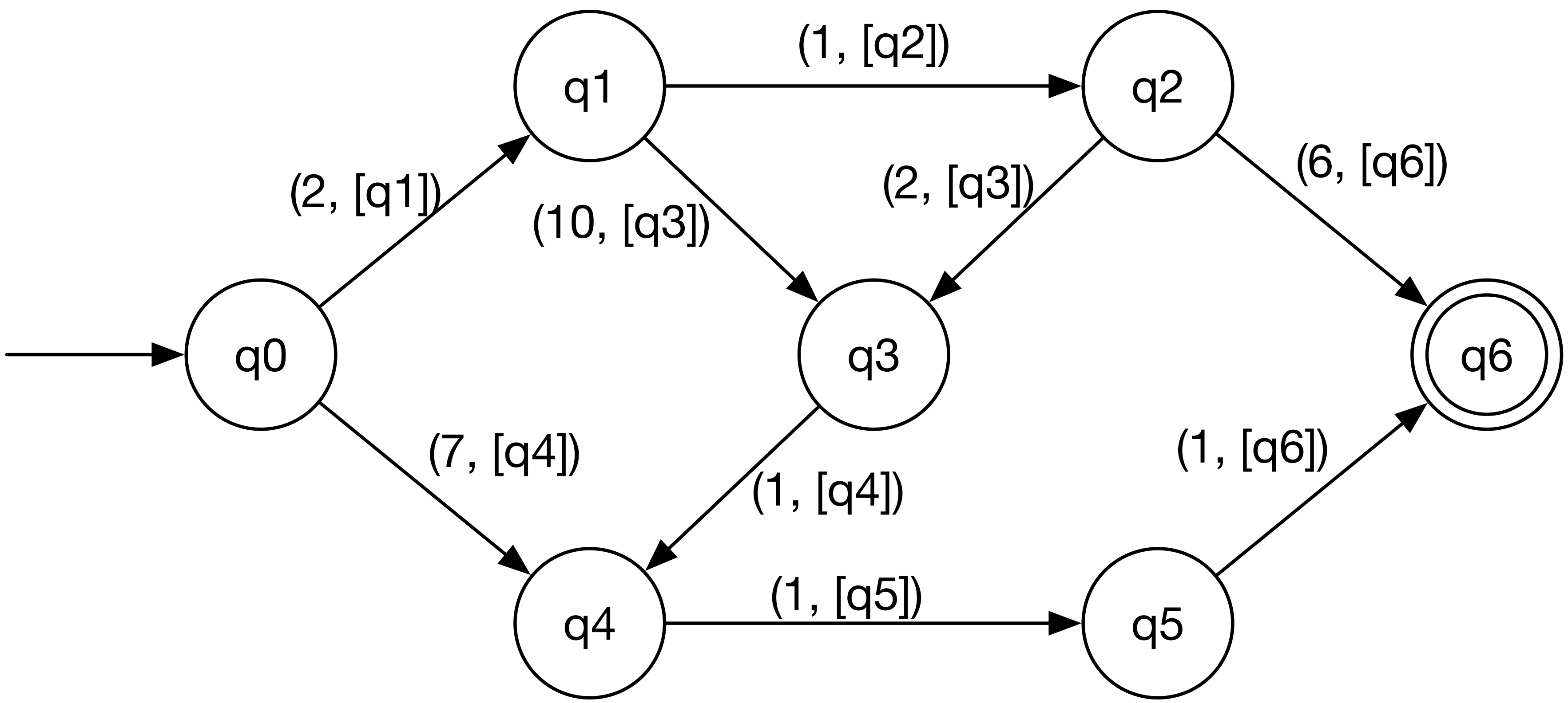


$$Q = (s, p)$$
$$s \in \mathbb{R} \cup \{+\infty\}$$
$$p \in \text{list of states}$$
$$\oplus = \min$$
$$\otimes = (+, \text{append})$$
$$\bar{0} = (+\infty, [])$$
$$\bar{1} = (0, [q_0])$$

$$\alpha(q_1) = (2, [q1]) \otimes \alpha(q_0) = (2 + 0, [q_0].[q_1]) = (2, [q_0, q_1])$$

q0	q1	q2	q3	q4	q5	q6
(0, [q0])	(2, [q0, q1])					

Best Path Semiring

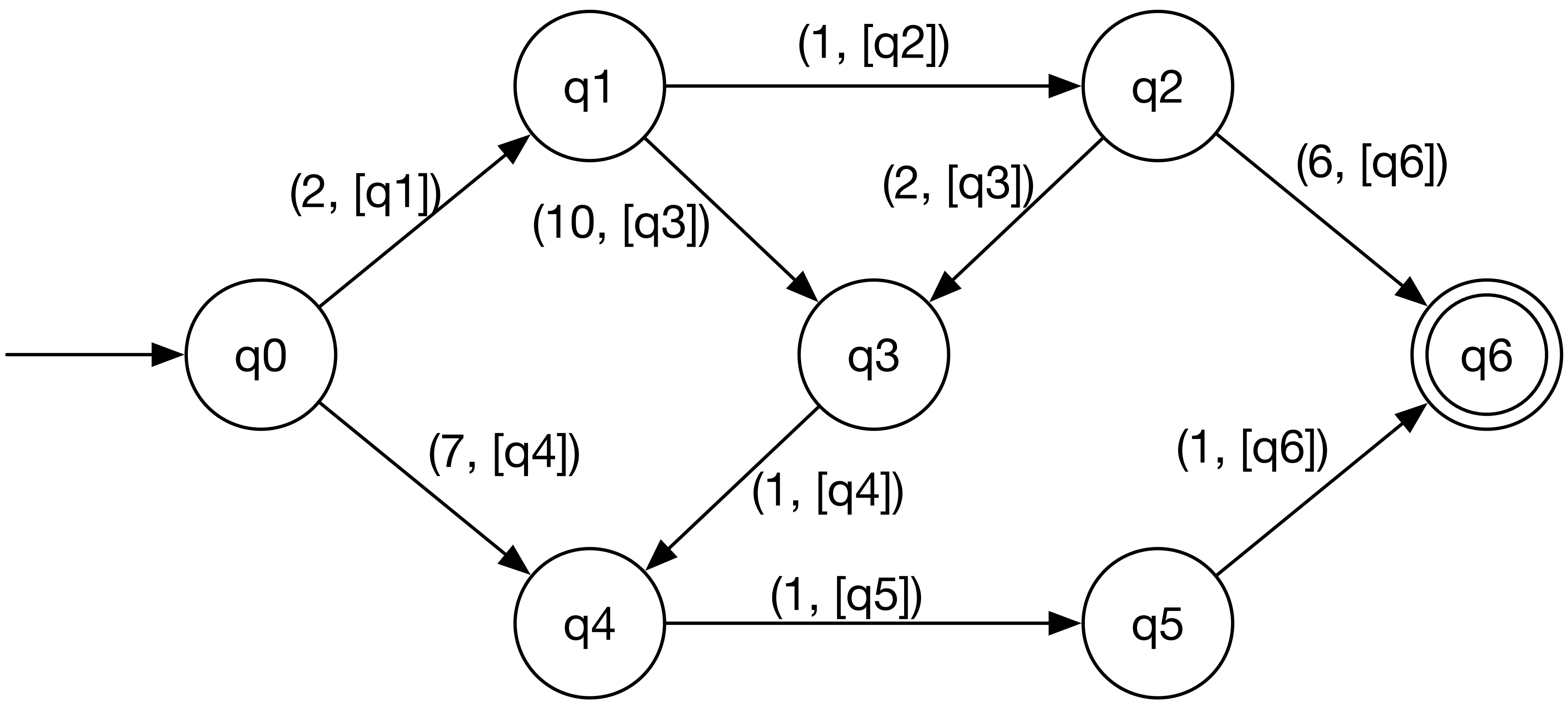


$$Q = (s, p)$$
$$s \in \mathbb{R} \cup \{+\infty\}$$
$$p \in \text{list of states}$$
$$\oplus = \min$$
$$\otimes = (+, \text{append})$$
$$\bar{0} = (+\infty, [])$$
$$\bar{1} = (0, [q_0])$$

$$\alpha(q_2) = (1, [q2]) \otimes \alpha(q_1) = (1 + 2, [q_0, q_1].[q_2]) = (3, [q_0, q_1, q_2])$$

q0	q1	q2	q3	q4	q5	q6
(0, [q0])	(2, [q0, q1])	(3, [q0, q1, q2])				

Best Path Semiring



$Q = (s, p)$

$s \in \mathbb{R} \cup \{+\infty\}$

$p \in$ list of states

$\oplus = \min$

$\otimes = (+, \text{append})$

$\bar{0} = (+\infty, [])$

$\bar{1} = (0, [q_0])$

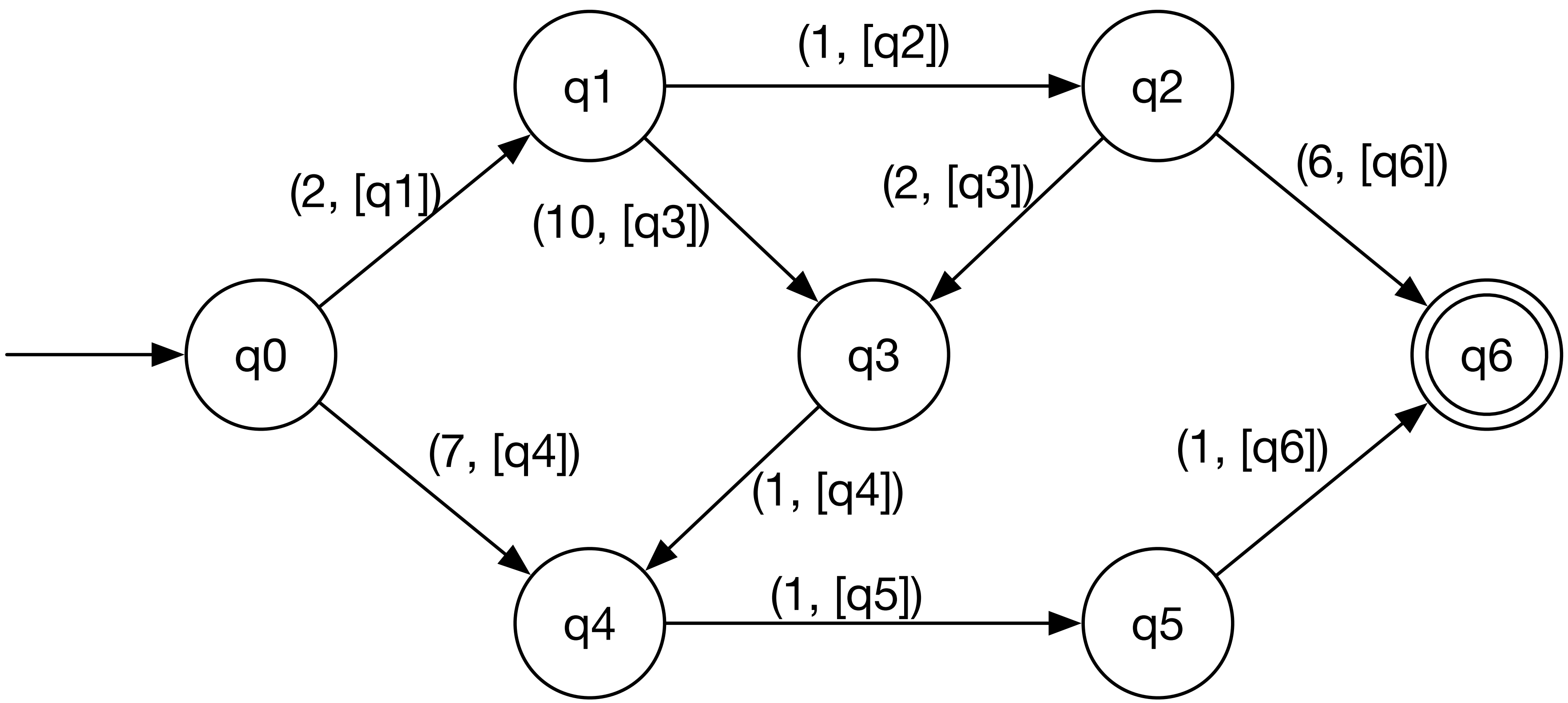
$$\alpha(q_3) = (10, [q_3]) \otimes \alpha(q_1) \oplus (2, [q_3]) \otimes \alpha(q_2)$$

$$\alpha(q_3) = \min((10 + 2, [q_0, q_1].[q_3]), (2 + 3, [q_0, q_1, q_2].[q_3]))$$

$$\alpha(q_3) = (5, [q_0, q_1, q_2, q_3])$$

q0	q1	q2	q3	q4	q5	q6
(0, [q0])	(2, [q0, q1])	(3, [q0, q1, q2])	(5, [q0, q1, q2, q3])			

Best Path Semiring

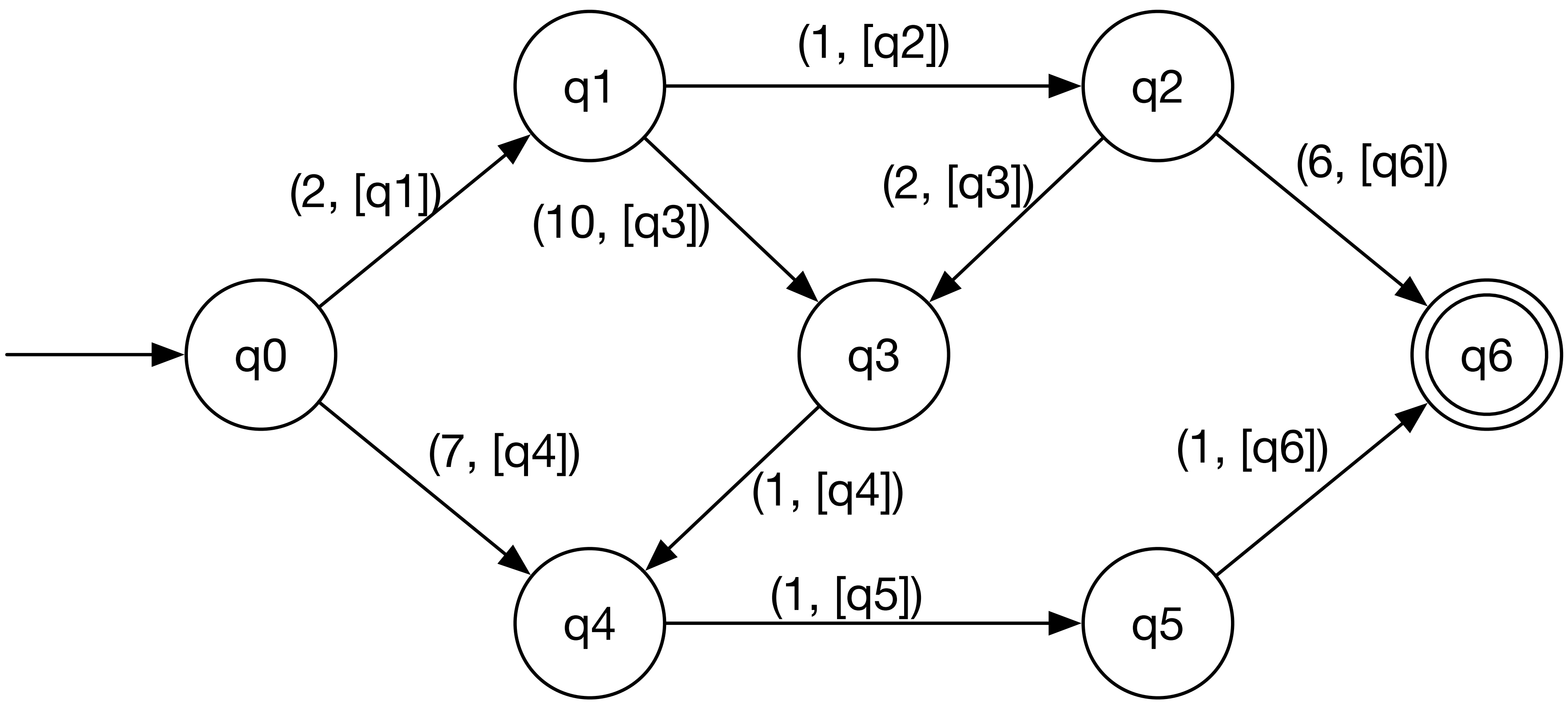


$$Q = (s, p)$$
$$s \in \mathbb{R} \cup \{+\infty\}$$
$$p \in \text{list of states}$$
$$\oplus = \min$$
$$\otimes = (+, \text{append})$$
$$\bar{0} = (+\infty, [])$$
$$\bar{1} = (0, [q_0])$$

$$\alpha(q_4) = (7, [q_4]) \otimes \alpha(q_0) \oplus (1, [q_4]) \otimes \alpha(q_3)$$
$$\alpha(q_4) = \min((7 + 0, [q_0].[q_4]), (1 + 5, [q_0, q_1, q_2, q_3].[q_4]))$$
$$\alpha(q_4) = (5, [q_0, q_1, q_2, q_3, q_4])$$

q0	q1	q2	q3	q4	q5	q6
(0, [q0])	(2, [q0, q1])	(3, [q0, q1, q2])	(5, [q0, q1, q2, q3])	(6, [q0, q1, q2,q3,q4])		

Best Path Semiring

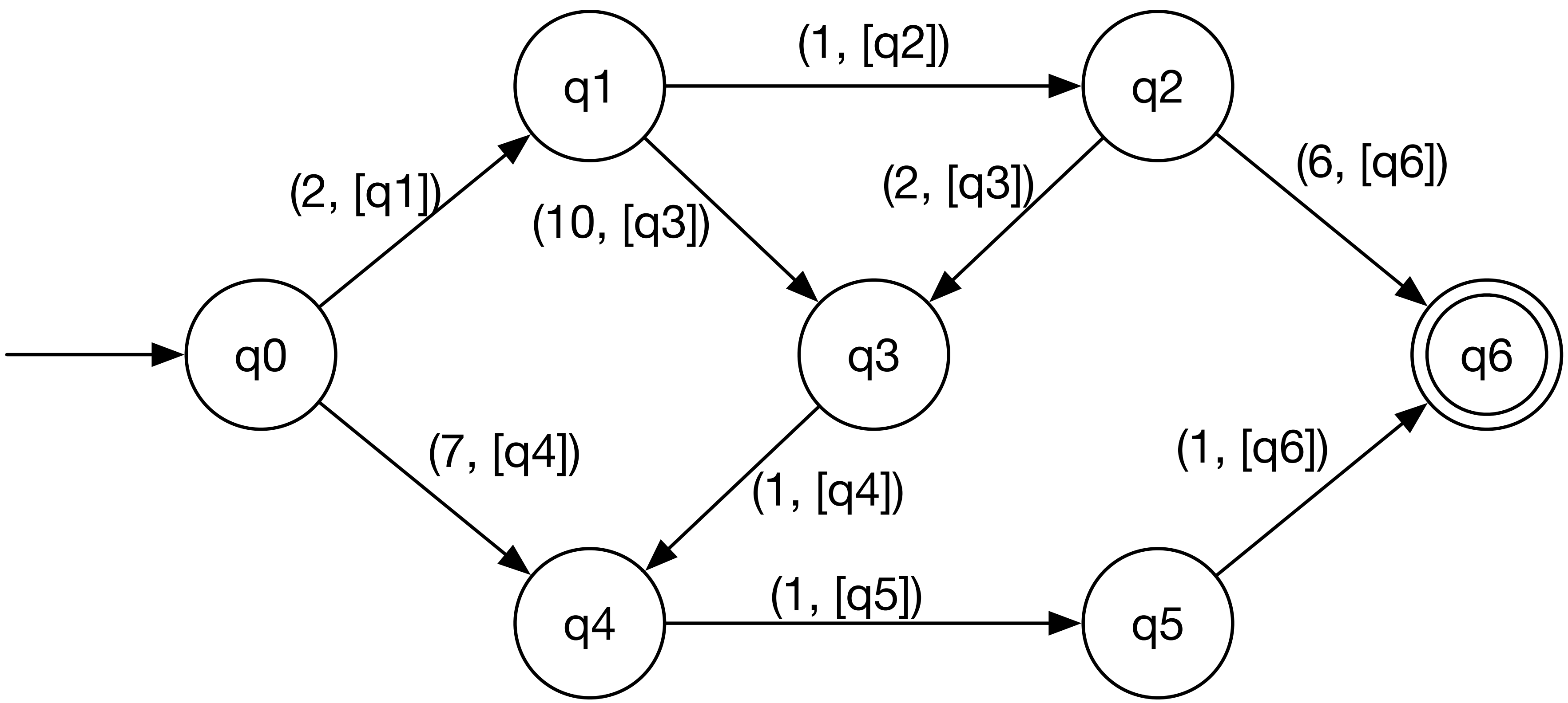


$Q = (s, p)$
$s \in \mathbb{R} \cup \{+\infty\}$
$p \in$ list of states
$\oplus = \min$
$\otimes = (+, \text{append})$
$\bar{0} = (+\infty, [])$
$\bar{1} = (0, [q_0])$

$$\alpha(q_5) = (1, [q_5]) \otimes \alpha(q_4)$$
$$\alpha(q_5) = (1 + 6, [q_0, q_1, q_2, q_3, q_4].[q_5])$$
$$\alpha(q_5) = (7, [q_0, q_1, q_2, q_3, q_4, q_5])$$

q0	q1	q2	q3	q4	q5	q6
(0, [q0])	(2, [q0, q1])	(3, [q0, q1, q2])	(5, [q0, q1, q2, q3])	(6, [q0, q1, q2,q3,q4])	(6, [q0, q1, q2, q3, q4, q5])	

Best Path Semiring

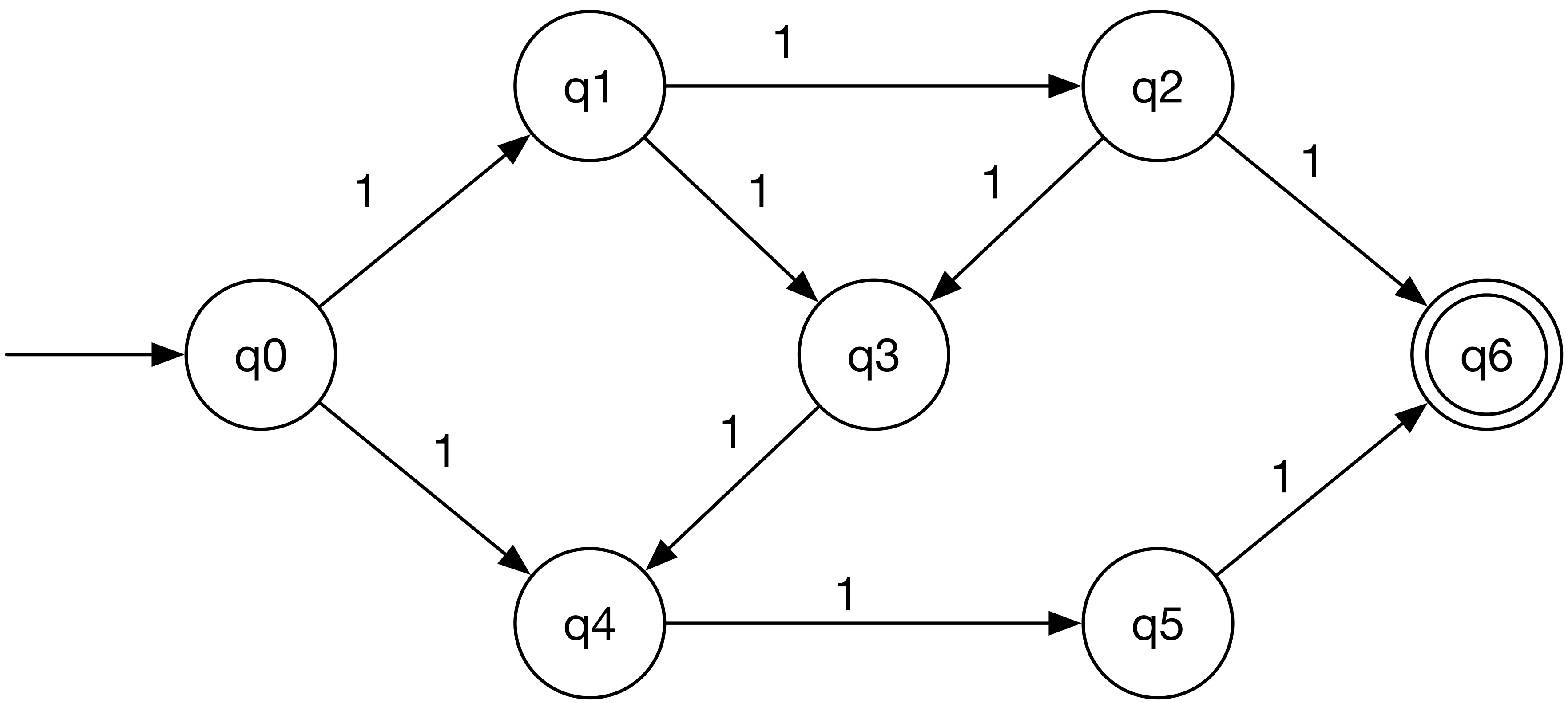


$$Q = (s, p)$$
$$s \in \mathbb{R} \cup \{+\infty\}$$
$$p \in \text{list of states}$$
$$\oplus = \min$$
$$\otimes = (+, \text{append})$$
$$\bar{0} = (+\infty, [])$$
$$\bar{1} = (0, [q_0])$$

$$\alpha(q_6) = (1, [q_6]) \otimes \alpha(q_5)$$
$$\alpha(q_6) = (1 + 7, [q_0, q_1, q_2, q_3, q_4, q_5].[q_6])$$
$$\alpha(q_6) = (8, [q_0, q_1, q_2, q_3, q_4, q_5, q_6])$$

q0	q1	q2	q3	q4	q5	q6
(0, [q0])	(2, [q0, q1])	(3, [q0, q1, q2])	(5, [q0, q1, q2, q3])	(6, [q0, q1, q2,q3,q4])	(7, [q0, q1, q2, q3, q4, q5])	(8, [q0, q1, q2, q3, q4, q5, q6])

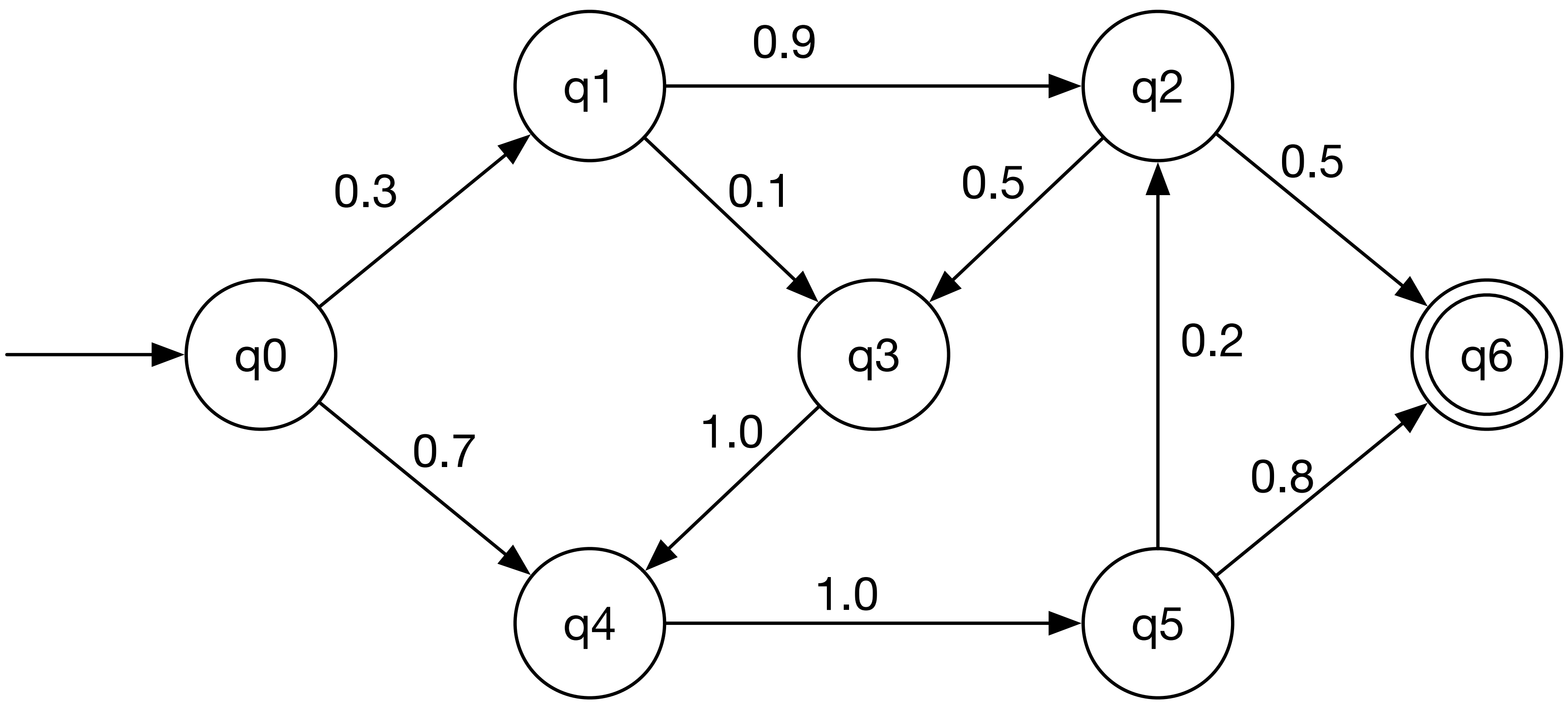
Example Exam Questions



$$Q = ??$$
$$\bigoplus = ??$$
$$\bigotimes = ??$$
$$\bar{0} = ??$$
$$\bar{1} = ??$$

Define a semiring such that $\alpha(q_i)$ is TRUE if there are an even number of paths from q_0 to q_i and FALSE otherwise.

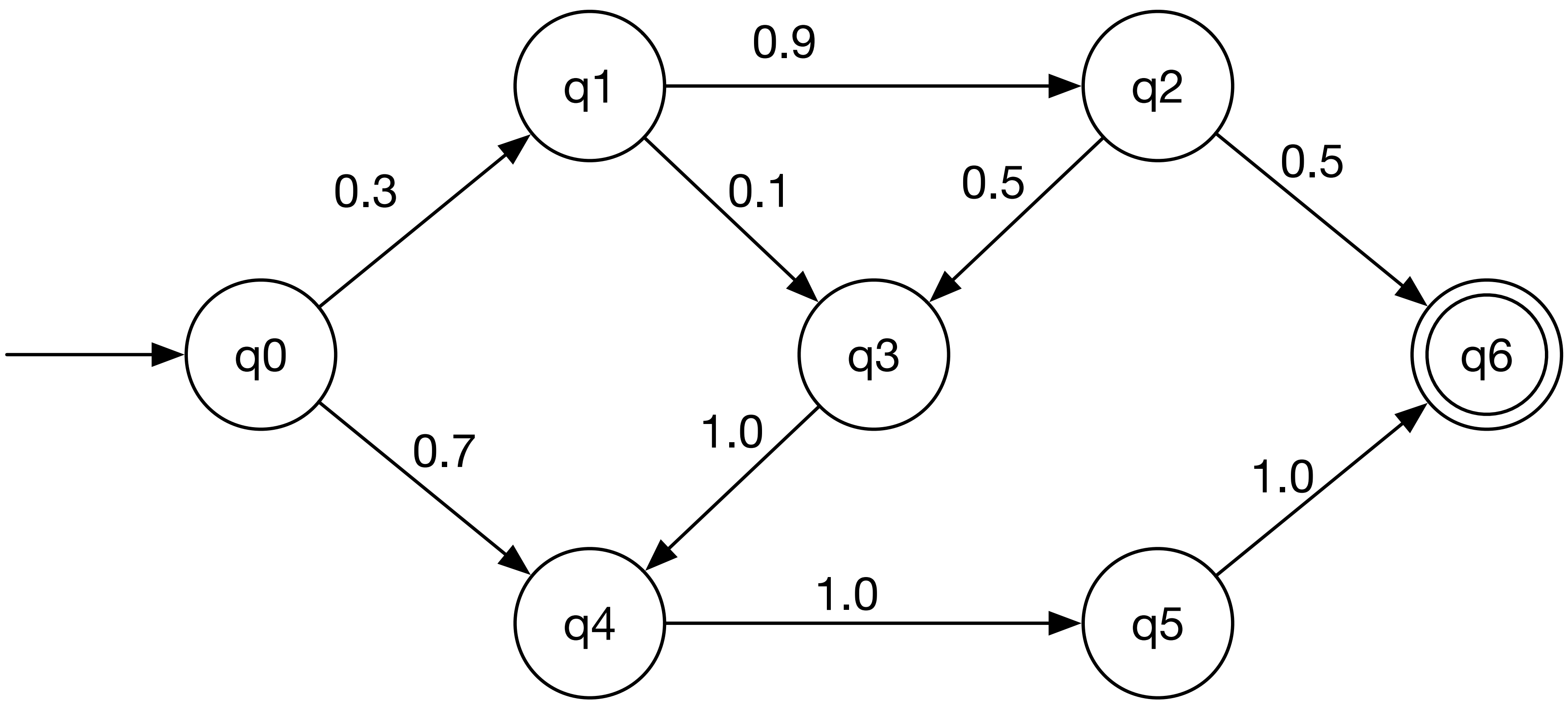
Example Exam Questions



$$Q = \mathbb{R}^{+}$$
$$\oplus = +$$
$$\otimes = \times$$
$$\bar{0} = 0$$
$$\bar{1} = 1$$

Use the probability semiring to calculate the probability that a path from q_0 to q_6 goes through each q_i. Mind the loop!

Example Exam Questions



$$Q = ??$$
$$\oplus = ??$$
$$\otimes = ??$$
$$\bar{0} = ??$$
$$\bar{1} = ??$$

Define a semiring such that $\alpha(q_i)$ gives $\mathbb{E}_{p(\text{path})}|\text{path}|$. That is the average number of edges you use if you randomly choose a path from q_0 to q_i .